Generalization of sum of positive integral powers of natural numbers

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Abstract

Sum of positive integral powers of first n natural numbers has been an interesting problem for many years. Mathematicians, students and research scholars have been attempting to crack this problem for decades. The primary objective of this talk is to generate a generalized result for an ancient interesting problem in the research field of Analytic Number Theory. That problem states that sum of k^{th} powers of first n- natural number coincides with a unique a polynomial of degree (k+1) in n over natural numbers. The existence and uniqueness of this polynomial are established using the principles of Linear Algebra. The innovative result derived here opens a way to write the formula for the sum of any positive integral power of first n- natural numbers.

Keywords*: Rank of a Matrix, Simultaneous Nonhomogeneous Linear Equations, Cramer's Rule, Binomial Coefficients, Coefficient matrix*

1.Introduction:

Thomas Harrlot (1560-1621) was the first mathematician who gave the generalized form of sum of positive integral powers of first n- natural numbers. Johann Faulhaber (1580-1635), a Germanmathematician, proposed formulas up to 17^{th} power and his work was considered a master piece at that time. However Johann Faulhaber failed to generalize his results. Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1705) were credited with the innovation of these results in explicit form. But Jacob Bernoulli (1654-1705) gave the most significant generalized formula explicitly.In 2012, DohyoungRyangand Tony Thompson, in their research article, generated a formula for sum of positive integralpowers of first n- natural numbers.Janet Beery, in 2010, in his paper, discussed the sum of positive integral powers of first n – natural numbers. Do Tan Si,in 2019,in research article proposed tables to compute Bernoulli numbers which are used in the generalization of sum of positive integral powers of first n- natural numbers

2. Existence and uniqueness of the generalized result:

Supposek,n are positive integers and

$$\sum_{r=1}^{n} r^k = 1^k + 2^k + 3^k + ...+ n^k = A_0 n^{k+1} + A_1 n^k + A_{20} n^{k-1} + ...A_k n + A_{k+1}$$

Here an assumption is being made that sum of k-th powers of first n –natural numbers coincides with a polynomial of degree (k+1) inn over natural numbers. Replace n by n+1

$$\sum_{r=1}^{n+1} r^k = 1^k + 2^k + 3^k + ...+ (n+1)^k$$
$$= A_0 (n+1)^{k+1} + A_1 (n+1)^k + A_2 (n+1)^{k-1} + ...A_k (n+1) + A_{k+1}$$

Subtracting the latter from the former

$$\begin{aligned}(n+1)^k = {} & A_0\left[(n+1)^{k+1} - n^{k+1}\right] \\ & + A_1\left[(n+1)^k - n^k\right] \\ & + A_2\left[(n+1)^{k-1} - n^{k-1}\right] \\ & + A_3\left[(n+1)^{k-2} - n^{k-2}\right] \\ & + ... \\ & + A_k\left[(n+1)^1 - n^1\right] \\ & + A_{k+1}\left[(n+1)^0 - n^0\right]\end{aligned}$$

$$\begin{aligned}& = A_0\left[(k+1)_{c_1} n^k + (k+1)_{c_2} n^{k-1} + ...+ (k+1)_{c_k} n + 1\right] \\ & + A_1\left[k_{c_1} n^{k-1} + k_{c_2} n^{k-2} + k_{c_3} n^{k-3} + ...+ k_{k-1} n + 1\right] \\ & + A_2\left[(k-1)_{c_1} n^{k-2} + (k-1)_{c_2} n^{k-3} + ...(k-1)_{c_{k-2}} n + 1\right] \\ & + A_3\left[(k-2)_{c_1} n^{k-3} + (k-2)_{c_2} n^{k-4} + ...(k-2)_{c_{k-3}} n + 1\right] \\ & + ... \\ & + A_{k-1}\left[2_{c_{1.}} n + 1\right] \\ & + A_k\end{aligned}$$

$$
\begin{aligned}
&= \left[A_0\left(k+1_{c_1}\right)\right]n^k + \left[A_0\left(k+1_{c_2}\right) + A_1 k_{c_1}\right]n^{k-1} \\
&+ \left[A_0\left(k+1_{c_3}\right) + A_1 k_{c_2} + A_2 (k-1)_{c_1}\right]n^{k-2} \\
&+ \left[A_0\left(k+1_{c_4}\right) + A_1 k_{c_3} + A_2 (k-1)_{c_2} + A_3 (k-2)_{c_1}\right]n^{k-3} \\
&+ \ldots\ldots \\
&+ \left[A_0\left(k+1_{c_k}\right) + A_1 k_{c_{k-1}} + A_2 (k-1)_{c_{k-2}} + A_3 (k-2)_{c_{k-3}} + \ldots\ldots\ldots + A_{k-1}.2_{c_1}\right]n \\
&+ \left[A_0\left(k+1_{c_{k+1}}\right) + A_1 k_{c_k} + A_2 (k-1)_{c_{k-1}} + A_3 (k-2)_{c_{k-2}} + \ldots\ldots\ldots + A_{k-1}.2_{c_{21}} + A_k .1_{c_1}\right]
\end{aligned}
$$

Applying binomial expansion for LHS and comparing the like powers of n^k one can get

$$k_{c_0} = A_0\left(k+1\right)_{c_1} \ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

$$k_{c_1} = A_0\left(k+1\right)_{c_2} + A_1 k_{c_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$k_{c_2} = A_0 (k+1)_{c_3} + A_1 k_{c_2} + A_2 (k-1)_{c_1} \ldots\ldots\ldots\ldots (3)$$

$$k_{c_3} = A_0\left(k+1\right)_{c_4} + A_1 k_{c_3} + A_2\left(k-1\right)_{c_2} + A_3\left(k-2\right)_{c_1} \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$\ldots$

$$k_{c_i} = A_0\left(k+1\right)_{c_{i+1}} + A_1 k_{c_{i-1}} + A_2\left(k-1\right)_{c_{i-2}} + \ldots\ldots A_i\left(k-i+1\right)_{c_1}$$

$$k_{c_{k-1}} = A_0\left(k+1\right)_{c_k} + A_1 k_{c_{k-1}} + A_2\left(k-1\right)_{c_{k-2}} + \ldots\ldots A_{k-1} 2_{c_1}$$

$$k_{c_k} = A_0\left(k+1\right)_{c_{k+1}} + A_1 k_{c_k} + A_2\left(k-1\right)_{c_{k-1}} + \ldots\ldots A_{k-1} 2_{c_2} + A_k 1_{c_1} \ldots\ldots\ldots\ldots\ldots\ldots (k+1)$$

The (k+1) equations constitute a system of linear non-homogenous equations in (k+1)-unknowns $A_0, \ldots.. A_k$. In matrix algebra notations this system is written as

$$
\begin{bmatrix}
(k+1)_{c_1} & 0 & 0 & . & 0 & 0 \\
(k+1)_{c_2} & k_{c_1} & 0 & . & 0 & 0 \\
(k+1)_{c_3} & k_{c_2} & k-1_{c_1} & . & 0 & 0 \\
. & . & . & . & . & . \\
(k+1)_{c_k} & k_{c_{k-1}} & k-1_{c_{k-2}} & . & 2_{c_1} & 0 \\
(k+1)_{c_{k+1}} & k_{c_k} & k-1_{c_{k-1}} & . & 2_{c_2} & 1_{c_1}
\end{bmatrix}
\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ . \\ . \\ A_k \end{bmatrix}
=
\begin{bmatrix} k_{c_0} \\ k_{c_1} \\ k_{c_2} \\ . \\ . \\ k_{c_k} \end{bmatrix}
$$

AX=B

$$\Delta = \begin{vmatrix} (k+1)_{c_1} & 0 & 0 & . & 0 & 0 \\ (k+1)_{c_2} & k_{c_1} & 0 & . & 0 & 0 \\ (k+1)_{c_3} & k_{c_2} & k-1_{c_1} & . & 0 & 0 \\ . & . & . & . & . & . \\ (k+1)_{c_k} & k_{c_{k-1}} & k-1_{c_{k-2}} & . & 2_{c_1} & 0 \\ (k+1)_{c_{k+1}} & k_{c_k} & k-1_{c_{k-1}} & . & 2_{c_2} & 1_{c_1} \end{vmatrix} \text{= Determinant of Coefficient Matrix}$$

$\Delta_0, \Delta_1, ...\Delta_k$ are obtained by replacing 1st,2nd…….(k+1)th columns by the column matrix

$$\begin{bmatrix} k_{c_0} \\ k_{c_1} \\ k_{c_2} \\ . \\ . \\ k_{c_k} \end{bmatrix}$$

$$\Delta = k+1_{c_1}.k_{c_1}.k-1_{c_1}.......2_{c_1}.1_{c_1}$$
$$= (k+1)! \neq 0 \; \textit{for any } k \in N$$

Since determinant of coefficient matrix is non zero and the number of equations is equal to number of unknowns, the above system of (k+1) –equations have a unique solution.

3. Formulas to compute the coefficients by Crammer's rule:

$$A_0 = \frac{\Delta_0}{\Delta}$$

$$A_1 = \frac{\Delta_1}{\Delta}$$

$$A_2 = \frac{\Delta_2}{\Delta}$$

.

.

$$A_k = \frac{\Delta_k}{\Delta}$$

By convention A_{k+1} is taken as 0.

As Δ is non -zero the above formulas prove the existence of the coefficients $A_0, A_1, A_2 .....A_k$

4. **Special Cases**

Case i)(For **k=1)**

$$\sum_{r=1}^{n} r^1 = 1^1 + 2^1 + 3^1 + ...+ n^1 = \sum n$$

$$1_{c_0} = A_0.2_{c_1} \Rightarrow 1 = A_0.2 \Rightarrow A_0 = \frac{1}{2}$$

$$1_{c_1} = \frac{1}{2}.2_{c_2} + A_1.1_{c_1} \Rightarrow A_1 = \frac{1}{2}$$

$$\therefore 1^1 + 2^2 + 3^1 + ...+ n^1 = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{n(n+1)}{2}$$

Case ii) (For k=2)

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + ...+ n^2 = \sum n^2 = A_0 n^3 + A_1 n^2 + A_2 n$$

$$2_{c_0} = A_0.3_{c_1} \Rightarrow A_0 = \frac{1}{3}$$

$$2_{c_1} = A_0.3_{c_2} + A_1.2_{c_1} \Rightarrow 2 = A_0 3 + A_1 2 \Rightarrow A_1 = \frac{1}{2}$$

$$2_{c_2} = A_0 3_{c_3} + A_1 2_{c_2} + A_2.1_{c_1}$$

$$1 = A_0 + A_1 + A_2 \Rightarrow A_2 = \frac{1}{6}$$

$$1^2 + 2^2 + 3^2 + ...n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n+1)(2n+1)}{6}$$

Case iii) (For k=3)

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + ...+ n^3 = A_0 n^4 + A_1 n^3 + A_2 n^2 + A_3 n$$

$$3_{c_0} = A_0 4_{c_1} \Rightarrow A_0 = \frac{1}{4}$$

$$3_{c_1} = A_0 4_{c_2} + A_1 3_{c_1} \Rightarrow A_1 = \frac{1}{2}$$

$$3_{c_2} = A_0 4_{c_3} + A_1 3_{c_2} + A_2 2_{c_1} \Rightarrow A_2 = \frac{1}{4}$$

$$3_{c_3} = A_0 4_{c_4} + A_1 3_{c_3} + A_2 2_{c_2} + A_3 1_{c_1} \Rightarrow A_3 = 0$$

$$\therefore 1^3 + 2^3 + 3^3 + ...+ n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = \frac{n^2(n+1)^2}{4}$$

Caseiv) (For k=4)

$$\sum_{r=1}^{n} r^4 = 1^4 + 2^4 + 3^4 + ...+ n^4 = A_0 n^5 + A_1 n^4 + A_2 n^3 + A_3 n^2 + A_4 n$$

$$4_{c_0} = A_0 5_{c_1} => A_0 = \frac{1}{5}$$

$$4_{c_1} = A_0 5_{c_2} + A_1 4_{c_1} \Rightarrow 4 = A_0.10 + A_1 4$$

$$\Rightarrow 4 = 2 + 4A_1 \Rightarrow A_1 = \frac{1}{2}$$

$$4_{c_2} = A_0.5_{c_3} + A_1.4_{c_2} + A_2.3_{c_1}$$

$$6 = 2 + 3 + A_2(3) \Rightarrow A_2 = \frac{1}{3}$$

$$4_{c_3} = A_0 5_{c_4} + A_1.4_{c_3} + A_2.3_{c_2} + A_3.2_{c_1}$$

$$4 = 1 + 2 + 1 + A_3(2) \Rightarrow A_3 = 0$$

$$4_{c_4} = A_0 + A_1 + A_2 + A_3 + A_4$$

$$1 = \frac{1}{5} + \frac{1}{2} + \frac{1}{3} + A_4 \Rightarrow A_4 = 1 - \left(\frac{1}{5} + \frac{1}{2} + \frac{1}{3}\right)$$

$$A_4 = -\frac{1}{30}$$

$$1^4 + 2^4 + 3^4 + ...n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{1}{30}n$$

Case v) (For k=5)

$$\sum_{r=1}^{n} r^5 = 1^5 + 2^5 + 3^5 + ...+ n^5 = A_0 n^6 + A_1 n^5 + A_2 n^4 + A_3 n^3 + A_4 n^2 + A_5 n$$

$$5_{c_0} = A_0\left(6_{c_1}\right) \Rightarrow A_0 = \frac{1}{6}$$

$$5_{c_1} = A_0 6_{c_2} + A_1 5_{c_1} \Rightarrow 5 \Rightarrow A_1 = \frac{1}{2}$$

$$5_{c_2} = A_0 6_{c_3} + A_1 5_{c_2} + A_2 4_{c_1} \Rightarrow A_2 = \frac{5}{12}$$

$$5_{c_3} = A_0 6_{c_4} + A_1 5_{c_3} + A_2 4_{c_2} + A_3 3_{c_1}$$

$$10 = \frac{5}{2} + 5 + \frac{5}{2} + 3.A_3 \Rightarrow A_3 = 0$$

$$5_{c_4} = A_0 6_{c_5} + A_1 5_{c_4} + A_2 4_{c_3} + A_3 3_{c_2} + A_4 2_{c_1}$$

$$5 = 1 + \frac{5}{2} + \frac{5}{3} + A_4 2 \Rightarrow 4 - \frac{5}{2} - \frac{5}{3} = 2A_4 \Rightarrow A_4 = -\frac{1}{12}$$

$$5_{c_5} = A_0 6_{c_6} + A_1 5_{c_5} + A_2 4_{c_4} + A_3 3_{c} + A_4 2_{c_2} + A_5 .1_{c_1}$$

$$1 = \frac{1}{6} + \frac{1}{2} + \frac{5}{12} - \frac{1}{12} + A_5 \Rightarrow A_5 = 0$$

$$\therefore 1^5 + 2^5 + 3^5 + ...+ n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

5. Simultaneous Non homogenous Liner equations:

$$k+1_{c_1}$$

$$k+1_{c_2} \quad k_{c_1}$$

$$k+1_{c_3} \quad k_{c_2} \quad k-1_{c_1}$$

$$k+1_{c_4} \quad k_{c_3} \quad k-1_{c_2} \quad k-2_{c_1}$$

$$k+1_{c_5} \quad k_{c_4} \quad k-1_{c_3} \quad k-2_{c_2} \quad k-3_{c_1}$$

(Table 1) (Binomial Triangle) (Coefficients)

Variables	RHS constants
A_0	k_{c_0}
A_1	k_{c_1}
A_2	k_{c_2}

A3	k_{c_3}
A_k	k_{c_k}

Table 2

From Tables (1) and (2) one can write simultaneous linear non-homogenous equations.

$$A_0(k+1)_{c_1} = k_{c_0}$$
$$A_0(k+1)_{c_2} + A_1.k_{c_1} = k_{c_1}$$
$$A_0(k+1)_{c_3} + A_1 k_{c_2} + A_2(k-1)_{c_1} = k_{c_2}$$
....
$$A_0(k+1)_{c_{k+1}} + A_1 k_{c_k} + A_2(k-1)_{c_{k-1}} + ...+ A_{k-1}2_{c_2} + A_k 1_{c_1} = k_{c_k}$$

It is interesting to note that

$$A_0 = \frac{1}{k+1}$$

$$A_1 = \frac{1}{2}$$

$$A_2 = \frac{k}{12}$$

$$A_3 = 0$$

$$A_4 = \frac{-k(k-1)(k-2)}{720} \quad \text{etc.}$$

From (k+1) thequation, one can observe thatsum of $A_0, A_1, A_2, ...A_k$ is 1.

$A_0+A_1+A_2+_{...}+A_k=1 \Rightarrow A_{k+1}=0$.

6. Conclusion and Future research:

The above conversation provides answers to the most interesting questions in the research field of Analytical Number Theory.

Q: Is the sum of positive integral powers of first n – natural numbers is a polynomial in n over natural numbers?

A: Yes

Q: Is such polynomial unique?

A: Yes.

The generalized result generated in this conversation certainly opens a way for young researchers and they can have a glance on these innovative ideas and may derive some more interesting results.

References:

1. DohyoungRyang, Tony Thompson (2012), “Sum of positive integral powers “August 2012 Mathematics Teacher 106 (1):71:77.

2. Janet Beery (2010), “Sum of powers of positive integers –conclusion”, Convergence,July 2010,MAA (Mathematics Teachers America)

3. Do Tan Si,”The PowersSums, Bernoullinumbers, Bernoulli Polynomials Rethinked”, Applied mathematics 10.03(2019):100-112, Scientific Research

4.Reznick,Bruce and Rouse j,”On the sums of two cubes”, International Journal of number theory,7(2011),1863-1882,MR 2854220

5.Mathematical Masterpieces, by Arthur Knoebel, et al., Springer (2007), ISBN 978-0-387-33060-0. An English translation of the Potestatum appears on pp 32-37 (from the French in the 1923 Grands Écrivains edition). Good discussion is had just before and after the Potestatum itself.

6."Pascal's Formula for the Sums of Powers of the Integers", by Carl B. Boyer, Scripta Mathematica 9 (1943), pp 237-244. Good historical background on antecedents to Pascal work on sums of powers. Contains an English translation of some of the Potestatum.

7.Pascal's Arithmetical Triangle, by A. W. F. Edwards, The Johns Hopkins University Press (2002, originally published 1987), ISBN 0-8018-6946-3.