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# Domatic Number of a Graph

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***Abstract***

***The domatic graph number is discussed in this chapter. Domatic was the word from the terms 'Dominance' and 'Chromatic' coined. Found domatic numbers for different type of graphs.***

## 1. Introduction

Graph Theory was developed in 1736 by Euler as a generalisation of the to the Koniberg bridges' famous problem solution.. Graph dominance has been thoroughly researched in the graph theory division. Berge in 1958 and Ore in 1962 formalised dominance as a theoretical field in graph theory. Ore was characterized by minimal dominant sets. Historically, from chess came the first domination – type issues. Several chess players were interested in the minimum number of queens in the 1850s, so that any square on the chess board includes either a queen (remember that a queen can move any number of squares on the chess board horizontally, vertically, or diagonally).

**Definition: 4.1**

A D-partition is a V(G) partition into a finite number of subsets. In every subset other than its own, each point in a subset is adjacent to at least one point in each subset.

**Definition: 4.2**

A D-partition of G is a partition of V(G) into a finite number of subsets. Each point in a subset in any sub-set other than its own is adjacent to at least one point in each sub-set.

**Proposition: 4.3**

For every graph G, $\mathbb{D}(G) \leq \delta(G) + 1$.

**Proof:**

If the domatic numbr of a graph is k , then every point in a dominating set of the partition is adjacent to atleast one point in each of other k - 1 dominating sets in the partition. In other words, each point must be adjacent to at least k-1 points, one in each dominant subset of the order k D-partition. This implies $\delta(G) \geq k - 1$.

Then $\delta(G) \geq \mathbb{D}(G) - 1$. Therefore $\mathbb{D}(G) \leq \delta(G) + 1$.

**Definition: 4.4**

A graph G is domatically full if $\mathbb{D}(G) = \delta(G) + 1$.

**Example:**

For any tree T, we have $\mathbb{D}(T) = \delta(G) + 1 = 2$. Therefore any tree is domatically full.

**Proposition: 4.5**

(a) $\mathbb{D}(K_n) = n$; $\mathbb{D}(K_n^*) = 1$

(b) $\mathbb{D}(K_n + G) = n + \mathbb{D}(G)$.

(c) (ore) $\mathbb{D}(G) \geq 2$ if and only if G has no isolated points.

(d) For any tree T with $n \geq 2$ points $\mathbb{D}(T) = 2$.

(e) For any $n \geq 3$, $\mathbb{D}(C_{3n}) = 3$, $\mathbb{D}(C_{3n+1}) = \mathbb{D}(C_{3n+2}) = 2$.

(f) For any $2 \leq r \leq s$, $\mathbb{D}(K_{r,s}) = r$.

**Proof:**

In the graph $K_n$, each point is adjacent to all other points. Then each point is a dominating set. There are n dominating sets. That is $\mathbb{D}(K_n) = n$. In $K_n*$, we have n-isolated points. We have only one dominating set.

Therefore $\mathbb{D}(K_n*) = 1$.

(a) In the join $(K_n + G)$ of two graphs, every point in $K_n$ is adjacent to every point of the graph G. Then each point in $K_n$ is a dominating set. By the definition of joint of two graphs, each dominating set of G is also a dominating set of $K_n + G$. The maximum number of dominating set in G is $\mathbb{D}(G)$. Therefore the maximum number of dominating set in $K_n + G$ is $n + \mathbb{D}(G)$. Hence $\mathbb{D}(K_n + G) = n + \mathbb{D}(G)$.

(b) By a theorem , if there is a set S of minimal dominating set of graph without isolated points, then V - S is also a dominating set. Therefore G has atleast two dominating sets.

Hence $\mathbb{D}(G) \geq 2$.

Conversely, suppose $\mathbb{D}(G) \geq 2$.. Suppose G has isolated points then G has only one dominating set. That is $\mathbb{D}(G) = 1$, a contradiction. Therefore, G has no isolated points.

(c)Since T has no isolated points, by proposition 4.5(c)

$\mathbb{D}(T) \geq 2$. (1)

For every graph G, $\mathbb{D}(G) \leq \delta(G) + 1$

For a tree T, $\delta(T) = 1$

Then $\mathbb{D}(T) \leq 2$. (2)

From (1) and (2), we get $\mathbb{D}(T) = 2$.

(d)For any cycle $C_n$, $\mathbb{D}(C_n) \leq \delta + 1 = 2 + 1 = 3$.

Consider $C_{3n.}$ Let 1,2,……………3n denote the points of $C_{3n}$.

$D_1 = \{1,4,7,……………..,3n-2\}$,

$D_2 = \{2,5,8,…………..,3n-1\}$ and

$D_3 = \{3,6,9,…………….3n\}$ are dominating sets. Hence $\mathbb{D}(C_{3n}) = 3$.

Next we prove that $\mathbb{D}(C_{3n+1}) = \mathbb{D}(C_{3n+2})$.

Consider $C_{3n+1}$. Each minimum dominating set consists of n + 1 points. $C_{3n+1}$ has at most two dominating sets.

Therefore $\mathbb{D}(C_{3n+2}) = 2$.

(e) Let $V = V_1 \cup V_2$ be the bipartition of V, and $|V_1| = r$, $|V_2| = s$ and $2 \leq r \leq s$. Each dominating set consists of one point from $V_1$ and another point from $V_2$. Hence each dominating set consists of at least two points. The maximum number of dominating sets in $K_{r,s}$ is r. Therefore $\mathbb{D}(K_{r,s}) = r$.

**Remark:**

$\mathbb{D}(K_{1,1}) = 2$. It does not satisfy $\mathbb{D}(K_{r,S}) = 2 \neq 1 = r$ if $r = s = 1$.

**Proposition: 4.6**

For every graph G having n points,

$Ɖ(G) + Ɖ(G^*) \leq n + 1.$

**Proof:**

By proposition 4.3, $Ɖ(G) \leq \delta(G) + 1$ (1)

and $Ɖ(G) \leq \delta(G) + 1 \leq \Delta(G^*) + 1$ (2)

Therefore $Ɖ(G) + Ɖ(G^*) \leq \delta(G) + \Delta(G^*) + 2$

$= \delta(G) + n - 1 - \delta(G) + 2 = n + 1$

$Ɖ(G) + Ɖ(G^*) \leq n + 1.$

**Theorem 4.7**

Let G have n points, then $Ɖ(G) + Ɖ(G^*) = n + 1$ if and only if $G = K_n$ or $K_n^*$

**Proof:**

Suppose $G = K_n$ or $K_n^*$. Then $Ɖ(K_n) = n$, $Ɖ(K_n^*) = 1$

If $G = K_n$, then $Ɖ(K_n) + Ɖ(K_n^*) = n+ 1.$

If $G = K_n^*$, then $Ɖ(K_n) + Ɖ(K_n^*) = Ɖ(K_n^*) + Ɖ(K_n^{**}) = 1 + n$.Conversely, suppose $Ɖ(G) + Ɖ(G^*) = n + 1$, we have to prove that, $G = K_n$ or $K_n^*$.

Suppose, contrary to the assertion that $G \neq K_n$ or $K_n^*$ have n points and $Ɖ(G) + Ɖ(G^*) = n + 1$.

**Case: (i)**

If $\delta(G) = 0$, then $Ɖ(G) = 1$.

G has an isolated point and G has a point u of degree n-1. Hence $G^* = K_1 + F$, where F has n - 1 points and $F \neq K_{n-1}$, otherwise $G^* = K_n$ which is not possible as $G \neq K_n$ which is not possible as $G \neq K_n$ or $K_n^*$.

As F is not complete graph, $Ɖ(F) < n - 1$

$Ɖ(G) + Ɖ(G^*) = Ɖ(G) + Ɖ(K_1 + F)$

$= Ɖ(G) + 1 + Ɖ(F)$ since $Ɖ(K_1 + F) = 1 + Ɖ(F)$

$= 1 + 1 + Ɖ(F)$ since $Ɖ(G) = 1$

$= 2 + Ɖ(F) < 2 + n - 1.$

$Ɖ(G) + Ɖ(G^*) < n - 1$, a contradiction.

**Case: (ii)**

$0 < \delta(G) < n / 2$ (1)

By assumption, $Ɖ(G^*) + Ɖ(G) = n + 1 - Ɖ(G)$

$\geq (n + 1) - \delta(G) + 1$, by proposition 4.3

Therefore, $Ɖ(G^*) \geq n - \delta(G)$

If all dominating sets in maximum D-partition of G* have at least two points,

Then $n \geq 2Ɖ(G^*) \geq 2(n - \delta(G)) = 2n - 2\delta(G)$ by (2)

$> 2n - n$

(Since $2\delta(G) < n$, $-2\delta(G) > -n$)

$n > n.$

This is a contradiction.

Hence some point v dominates G*.

Therefore, $deg_G(v) = n-1$ and $deg_G(v) = 0$.
Hence $\delta(G) = 0$.
This is contradiction to (1)

**Case: (iii)**

$n / 2 \leq \delta(G) < n - 1$ (3)

In this case, since for every graph of n points,
$\delta(G) + \delta(G^*) \leq n - 1$
$\delta(G) \leq n - 1 - \delta(G)$
$\leq n - 1 - (n/2)$
$\leq (n/2) - 1$ by (3)
If $\delta(G^*) = 0$, apply case(i) to G, otherwise apply case(ii) to G*.
Hence proved the statement .

**Theorem: 4.8**

For every graph G with n points, $\text{Ɗ}(G) + \gamma(G) \leq n + 1$, with equality if and only if $G = K_n$ or $K_n^*$.

**Proof:**

If $G = K_n$, $\text{Ɗ}(G) = n$ and $\gamma(G) = 1$.
If $G = K_n^*$, then $\text{Ɗ}(G) = 1$ and $\gamma(G) = n$.
If $G = K_n$ or $K_n^*$, then $\text{Ɗ}(G) + \gamma(G) = n + 1$.
If $\text{Ɗ}(G) + \gamma(G) = n + 1$, we have to prove that $G = K_n$ or $K_n^*$
If G has n points and $G \neq K_n$ or $K_n^*$, then we shall prove that
$\text{Ɗ}(G) + \gamma(G) < n + 1$.
If G has n points and maximum degree $\Delta$, then $\gamma(G) \leq n - \Delta(G)$
Hence $\gamma(G) \leq n - \delta(G)$ (1)
Since $\delta(G) \leq \Delta(G)$, $-\Delta(G) \leq -\delta(G)$
We claim that this latter inequality or $\text{Ɗ}(G) \leq \delta(G) + 1$ is strict (2)
That is, $\gamma(G) = n - \delta(G)$ and $\delta(G) = \text{Ɗ}(G) - 1$.
Suppose not, then $\gamma = n - \delta = n - (\text{Ɗ}(G) - 1)$ Since $\text{Ɗ}(G) = \delta(G)+1$

$\gamma = n - \text{Ɗ}(G) + 1$

By definition of $\gamma(G)$ and $\text{Ɗ}(G)$, we have

$\text{Ɗ}(G) \leq n/ \gamma(G)$

$\text{Ɗ}(G) \leq n/(n - \text{Ɗ}(G) + 1)$ by (3)

$\text{Ɗ}(G) (n-\text{Ɗ}(G)+1) \leq n..$

$\text{Ɗ}(G)n - (\text{Ɗ}(G))^2 + \text{Ɗ}(G) - n \leq 0$

$\text{Ɗ}(G)(n - \text{Ɗ}(G)) - (n - \text{Ɗ}(G)) \leq 0$

$(\text{Ɗ}(G) - 1) (n - \text{Ɗ}(G)) \leq 0$ (4)

$\text{Ɗ}(G) \geq 1$, $n \geq \text{Ɗ}(G)$ this implies that $\text{Ɗ}(G) - 1 \geq 0$,
$n - \text{Ɗ}(G) \geq 0$.

(Ɖ(G) – 1) (n – Ɖ(G)) $\geq$ 0

This implies Ɖ(G) – 1 = 0 or n – Ɖ(G) = 0

This implies n = Ɖ(G) or Ɖ(G) = 1.

If n = Ɖ(G), then G = $K_n$. If Ɖ(G) = 1, then G has an isolated point v.

As G $\neq$ $K_n$*, we have, G – v $\neq$ $K_{n-1.}$ This implies that y(G-v) $\leq$ n - 1 and y(G) < n.

In this case, since $\delta = 0$, the inequality y(G) $\leq$ n - $^{\delta}$(G) is strict. Thus one of the inequalities y(G) $\leq$ n - $^{\delta}$(G) or Ɖ(G) $\leq$ $^{\delta}$(G) + 1 is strict implies that y(G) + Ɖ(G) < n- $^{\delta}$(G) + $^{\delta}$(G) + 1 = n + 1. Hence Ɖ(G) + y(G) $\leq$ n + 1.

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