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Combination of Triangular and Quadrilateral Edge Element for the Eigenvalue Analysis of Electromagnetic Wave Propagation

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Abstract- Eigenvalue analysis of electromagnetic wave propagation for circular domain with and without the presence of the crack is done. Combination of triangular and quadrilateral edge elements are used to discretize the circular domain. Conversion algorithm is used to convert the nodal element data to edge element data. We compared the results of both the domains obtained with edge element along with the analytical and nodal element values. Convergence study is also performed to know the effectiveness of both lower and higher order edge elements for normal and cracked circular domains. For the cracked circular domain, nodal elements fail to capture the singular eigenvalue which is well captured by the edge element.

Keywords – FEM, Edge element, Eigenvalue problem, Electromagnetic wave propagation

I. INTRODUCTION

Finite element method (FEM) has been used to solve electromagnetic problems. These problems include the eigenvalue analysis of electromagnetic domains [1-3]. In order to implement the FEM technique, the domain has to be discretized with either nodal or edge elements. But the following drawbacks limits the application of nodal based FEM in electromagnetic analysis [4-13]:

1. Due to improper continuity condition at the material interfaces occurrence of spurious solutions in eigenvalue problems.
2. Nodal based elements fails to capture singular eigenvalues in the domains where the sharp corners and edges are present.
3. Due to nodal continuity requirement, these elements cannot give direct solutions in terms of electric and magnetic field variables. We have to follow potential formulations.
4. Due to the requirement of tangential continuity and normal discontinuity across material interfaces for vector fields, special type of elements are required.

When edge elements are used in FEM these drawbacks can be avoided. These elements are constructed by curl and divergence conforming spaces. Whitney introduced these elements in the field of FEM in electromagnetics. In this element [4-17] electric fields are along the edge of the element satisfying required tangential continuity.

The rest of the paper is organized as follows. Discussion about the conversion algorithm used to convert nodal information of finite element to edge element is placed in section II. In section-III we presents the eigenvalues for the circular domain with and without the presence of crack for both lower and higher order edge elements. We also performs convergence study of the lower order and higher order of combination of triangular and quadrilateral edge elements. Concluding remarks are given in section IV.

II. Conversion of Nodal data to Edge Data

In nodal based FEM each element is formed with the nodes. But in edge based FEM each element is formed by connecting the edges. Each edge of the edge element is formed by joining the two nodes of the element. In FEM implementation edge information and edge connectivity data are required. But most of the available mesh generator are designed for nodal elements. So, a separate conversion algorithm is required which converts the nodal information of the element to edge information.

In this conversion algorithm the outer loop runs over total number of discretized elements. The inner most loop runs over the total number of edges of each element. Inside the inner most loop, a subroutine supplies the two end nodes based on local nodal connectivity and local edge connectivity of the element to form new edge. With the help of *nodeedge* array checking is performed whether any edge is already assigned between the current local end nodes (starting and end nodes). This *nodeedge* array stores the global edge number and its corresponding other end node in odd columns and even columns respectively. If no edge is found then existing last global edge number is incremented by 1. This number is assigned to current local edge and updated in the edge connectivity array. Then the loop runs for the next local edge. This process continues for all the local edges and then jump to the element loop for next element. Along with the edge data this algorithm stores the directional information of the edges and number of connecting edges with each global node of the finite elements domain.

III. NUMERICAL ANALYSIS AND RESULTS

3.1 Eigen Analysis

The governing differential equation for harmonic excitation with no external load condition (current density is zero) can be given as [3]

$$\nabla\times\left(\frac{1}{\mu_r}\nabla\times\boldsymbol{E}\right)-k_0^2\varepsilon_r\boldsymbol{E}=0 \qquad (1)$$

Where $k_0 = \omega/c$ is the wave number in vacuum, ω is the excitation frequency (rad/sec), $c = 1/\sqrt{\mu_0\varepsilon_0}$ is the speed of light in vacuum (m/sec), $\boldsymbol{E}$ is the electric field (v/m), $\mu_r = \mu/\mu_0$ is the relative permeability and $\varepsilon_r = \varepsilon/\varepsilon_0$ is the relative permittivity and ε_0 and μ_0 are the permittivity and permeability for vacuum respectively. The equation (1) is used to solve the eigenvalue problems for circular domains with crack and without crack. In order to validate the edge elements implemented, we have considered 2D eigenvalue problem from [18, 19]. We have assumed the properties $\mu_r = \varepsilon_r = 1.0$ and on the surface of the domain conducting boundary condition ($\boldsymbol{E} \times \boldsymbol{n} = 0$) is applied. Here $\boldsymbol{n}$ is the normal vector to the surface of the domain.

3.2 Circular domain with perfectly conducting surfaces

In order to perform the modal analysis, a circular domain of unit radius with perfectly conducting surfaces is considered. We consider two different cases for meshing the domain. In one case we discretize the domain with 3-edge triangle (T3) and 4-edge quadrilateral (Q4) elements. Here we use uniform 30 x 70 meshes along r and θ directions. In another case we mesh the domain with 8-edge triangular element (T8) and 12-edge quadrilateral element (Q12). In this case we use uniform 20 x 40 meshes along r and θ directions. Figure 1 shows the discretized domain with nodal triangular and quadrilateral elements. B6, B9 are the conventional 6-node triangular element and 9-node quadrilateral element respectively. This mesh data is converted to edge element data with the help of node to edge conversion algorithm. Triangular elements are used for meshing around the origin in a layer and the remaining domain with quadrilateral elements.

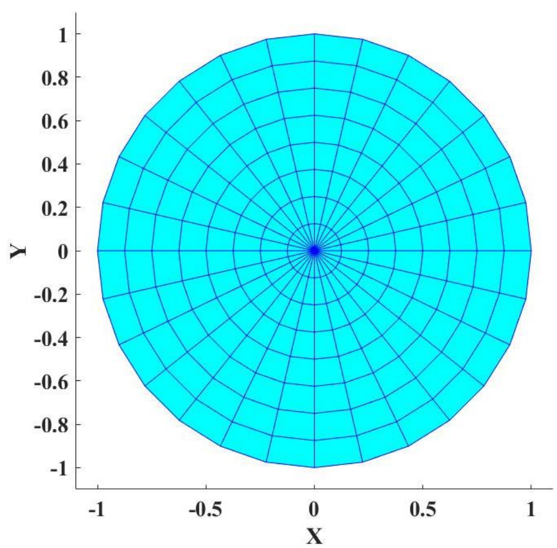

Figure 1. Finite element mesh of circular domain

Table -1 k_0^2 on the circular domain (bracketed values shows the multiplicity)

Analytical	Nodal element	Edge element	
	(B6/B9)	(T3/Q4)	(T8/Q12)
3.391122(2)	3.388839(2)	3.425702(2)	3.384698(2)
9.329970(2)	9.317991(2)	9.529880(2)	9.329397(2)
14.680392(1)	14.667718(1)	14.799790(1)	14.724344(1)
17.652602(2)	17.607352(2)	18.346646(2)	17.662205(2)
28.275806(2)	28.155200(2)	28.654137(2)	28.332827(2)
28.419561(2)	28.372027(2)	30.104019(2)	28.343553(2)
41.158640(2)	40.883369(2)	45.173345(2)	41.404574(2)
44.970436(2)	44.839236(2)	45.556597(2)	44.974676(2)
49.224256(1)	49.084169(1)	49.647942(1)	49.469042(1)
56.272505(2)	55.722110(2)	60.581974(2)	56.272505(2)
64.240225(2)	63.964270(2)	65.750854(2)	64.271152(2)
Number of computed zeros			
-	320	709	829

Table 1 show the eigenvalues of full circular domain discretized with the combination of 3-edge triangular and 4-edge quadrilateral elements (T3/Q4) and the combination of 8-edge triangular and 12-edge quadrilateral elements (T8/Q12) along with the number of computed zeros. We tried to compare edge element results along with the analytical results reported in [18, 19] and conventional nodal elements results from [20, 21]. For both the nodal elements and proposed elements results are in good match with the analytical results.

3.3 *Cracked circular domain*

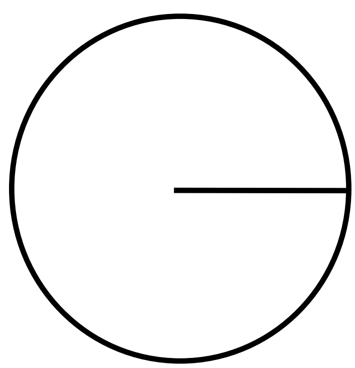

Figure 2. Cracked circular domain

In the cracked circular domain of unit radius where the crack runs from the centre to the periphery of the domain as shown in Figure 2. The crack is modeled by using 'double noding' along the crack. We use 30 x 70 finite elements (T3/Q4) along the along r and θ directions to discretize the domain in one case. In another case we use 20 x 40 T8/Q12 finite elements for meshing. The nodal finite element information of the discretized domain is generated with the help of finite element mesh generator HyFem. Then by using the conversion algorithm nodal data is converted to edge element information.

Table -2 k_0^2 on the cracked circular domain

Analytical	Nodal element	Edge element	
	(B6/B9)	(T3/Q4)	(T8/Q12)
1.358390	-	1.363529	1.299520
3.391122	4.391601	3.425756	3.384717
6.059858	6.094458	6.146601	6.054111
-	6.431327	-	-
9.329970	9.315878	9.529880	9.329397
13.195056	13.170984	13.588555	13.201275
Number of computed zeros			
-	913	759	1080

Table 2 show the eigenvalues of cracked circular domain discretized with the T3/Q4 elements as well as T8/Q12 elements along with the number of computed zeros. We compared the edge element results along with the analytical solutions and nodal element results. Edge based finite element method results are in good agreement with the analytical eigenvalues. But results obtained from nodal frame work have spurious nonzero eigenvalues of 6.4315878 and one singular eigenvalue is not predicted as shown in Table 2.

3.4 Convergence study

We performed the convergence analysis to understand the efficiency and performance of the edge elements (T3/Q4, T8/Q12). Here, T3 is the 3-edge triangular element and Q4 is the 4-edge quadrilateral element. In order to perform the convergence study, we have taken 7 different meshes for circular domain with perfectly conducting boundary without the presence of crack. For cracked circular domain with the conducting surface we have considered 5 different meshes. The percentage of error is plotted against the total number of degrees of freedom (dof) or total number of equations. This is to correlate the percentage of error with the computational cost. Fig. 3 and Fig. 4 show the convergence plots for the combination of triangular and quadrilateral elements of both lower and higher order elements for both the circular domains. Percentage of error is calculated by using the equation (2).

$$\text{Error}(\%) = \frac{\text{Numerical value} - \text{Analytical value}}{\text{Analytical value}} \times 100 \qquad (2)$$

3.4.1 Combination of 3-edge triangular and 4-edge quadrilateral elements (T3/Q4)

Full circular domain having crack (Figure 2) with conducting surfaces is discretized with T3/Q4 elements. For 5 different meshes (96, 520, 750, 1200, 1600 elements) total no. of equations i.e. free degree of freedoms are 183, 1035, 1496, 2391 and 3201 respectively. Figure 3 shows convergence plot for the first five eigen frequencies. It can be observed that percentage of error for all the five eigen frequencies lies below 12%. But as the mesh refinement increases percentage of error reduces to less than 2%.

3.4.2 *Combination of 8-edge triangular and 12-edge quadrilateral elements* (T8/Q12)

T8/Q12 elements are used to mesh the circular domain without crack (Figure 1) to perform the convergence analysis. Here the domain is having perfectly conducting surfaces. For different mesh sizes (50, 270, 400, 500, 600, 700, 800 elements) of the circular domain 380, 2040, 3120, 3920, 4720, 5520 and 6320 are the total degrees of freedom respectively. Percentage of error is plotted against the total no. of equations and figure 4 shows the convergence study plot for the first five eigen frequencies of the circular domain. It can be interpreted that as the mesh refinement increases percentage of error with analytical value is decreasing. Moreover the percentage of for all the eigen frequencies lies below 2% for the finer mesh refinement.

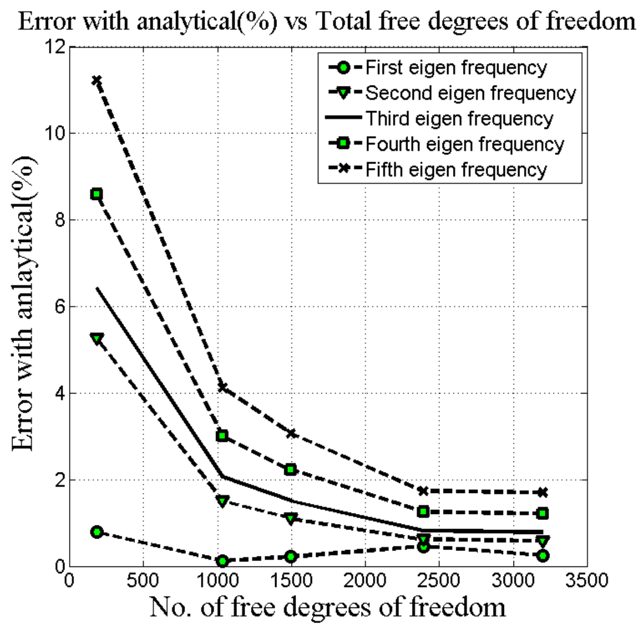

Figure 3. Convergence analysis of cracked circular domain discretized with 3-edge triangular and 4-edge quadrilateral elements.

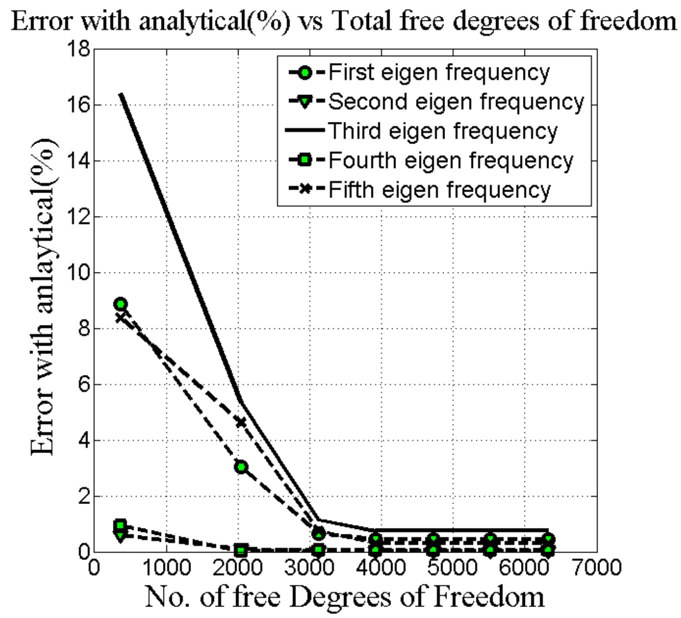

Figure 4. Convergence analysis of circular domain discretized with 8-edge triangular and 12-edge quadrilateral elements

IV. CONCLUSIONS

In this work we compared nodal elements with edge elements while triangular and rectangular elements are combined for solving some electromagnetic eigenvalue problems. The edge elements yield accurate eigenvalues along with the correct multiplicity. But nodal elements fail to capture the singular eigenvalue and there is one spurious eigenvalue for cracked circular domain. These shortcomings are not present in edge elements. From the convergence study it can be concluded that for finer mesh refinement percentage of error is decreasing and lies below 2%. Proposed combination of edge elements are performing better for both lower and higher order elements.

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