Stability of Additive Non-Additive Functional Equation

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Abstract

In this paper, the authors introduce and investigate the general solution and generalized Ulam-Hyers stability of a generalized young type of additive-quadratic functional equation

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$=\frac{1}{a}[q(x)-q(-x)]+\frac{1}{b}[q(y)-q(-y)]+\frac{1}{c}[q(z)-q(-z)]$$
$$+2\left(\frac{1}{a}\right)^2[q(x)+q(-x)]+2\left(\frac{1}{b}\right)^2[q(y)+q(-y)]+2\left(\frac{1}{c}\right)^2[q(z)+q(-z)]$$

where n is a positive integer with a,b,c are not equal to zero, in Banach Space and Banach Algebras using direct method approach

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1. Introduction and Preliminaries

Over the last seven decades, the above problem was talked by numerous authors and its solutions via various forms of functional equations like, additive, quadratic, cubic, quartic mixed type functional equations were discussed. We refer the interested readers for more information on such problems to the monographs of [4,6,9,11,13].

One of the most famous functional equation is the additive functional equation

$$f(x+y)=f(x)+f(y).$$

(1)

In 1821, it was first solved by A.L. Cauchy in the class of continuous real - valued functions. It is often called Cauchy additive functional equation in honor of A.L. Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1) is called an additive function.

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2}$$

and therefore, the equation (2) is called quadratic functional equation. The Hyers - Ulam stability theorem for the quadratic functional equation (2) was proved by F.Skof for the functions $f : E_1 \to E_2$ where E_1 is a normed space and E_2 be a Banach space.

The solution and stability of the following mixed type additive-quadratic functional equations

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 2f(2x) - 4f(x) \tag{3}$$

$$f(x+2y) + f(x-2y) + 8f(y) = 2f(x) + 4f(2y) \tag{4}$$

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f\left(x_i\right) = \sum_{1 \le i < j \le n} f\left(x_i + x_j\right) \tag{5}$$

Motivated by the above findings in this paper, we introduce and investigate that the general solution and generalized Ulam-Hyers stability of a generalized n-type Additive Quadratic functional equation of the form

$$\begin{aligned} &q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right) + q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right) + q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right) + q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right) \\ &= \frac{1}{a}[q(x) - q(-x)] + \frac{1}{b}[q(y) - q(-y)] + \frac{1}{c}[q(z) - q(-z)] \\ &+2\left(\frac{1}{a}\right)^2 [q(x) + q(-x)] + 2\left(\frac{1}{b}\right)^2 [q(y) + q(-y)] + 2\left(\frac{1}{c}\right)^2 [q(z) + q(-z)] \end{aligned} \tag{6}$$

where $\frac{1}{a}$ is a positive integer with $\frac{1}{a} \neq 0$, in Banach spaces and Banach Algebras using direct and fixed point methods.

2. General Solution of The Functional Equation (6): When q is Odd

In this section, the general solution of the functional equation (6), for odd case is discussed. Throughout this section, let us consider X and Y to be real vector spaces.

Theorem 2.1. If an odd mapping $q : X \to Y$ satisfies the functional equation

$$q(x+y) = q(x) + q(y) \tag{7}$$

for all $x, y \in X$, if and only if $q : X \to Y$ satisfies the functional equation

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right) + q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right) + q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right) + q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$

$$= \frac{1}{a}[q(x) - q(-x)] + \frac{1}{b}[q(y) - q - y)] + \frac{1}{c}[q(z) - q(-z)]$$

$$+2\left(\frac{1}{a}\right)^2 [q(x) + q(-x)] + 2\left(\frac{1}{b}\right)^2 [q(y) + q(-y)] + 2\left(\frac{1}{c}\right)^2 [q(z) + q(-z)]$$

(8)

for all $x, y, z \in X$.

Proof. Let $q : X \to Y$ satisfies the functional equation (7). Setting (x, y) by $(0, 0)$ in (7), we get $q(0) = 0$. Replacing (x, y) by (x, x) and $(x, 2x)$ respectively in (7), we obtain

$q(2x) = 2q(x)$ and $q(3x) = 3q(x)$

(9)

for all $x \in X$. In general for any positive integer a, we have

$q(ax) = aq(x)$

(10)

for all $x \in X$. It is easy to verify from (10) that

$q(a^2x) = a^2q(x)$ and $q(a^3x) = a^3q(x)$

(11)

for all $x \in X$. Replacing (x, y) by $\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$ in (8) and using (7), (10) and (11), we get

$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right) = \left(\frac{1}{a}\right)q(x) + \left(\frac{1}{b}\right)q(y) + \left(\frac{1}{c}\right)q(z)$

(12)

for all $x, y, z \in X$. Again replacing z by $-z$ in equation (12) and using oddness of q, we obtain

$g\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right) = \left(\frac{1}{a}\right)g(x) + \left(\frac{1}{b}\right)g(y) - \left(\frac{1}{c}\right)g(z)$

(13)

for all $x, y, z \in X$. Also replacing y by $-y$ in (12) and using oddness of q, we get

$q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right) = \left(\frac{1}{a}\right)q(x) - \left(\frac{1}{b}\right)q(y) + \left(\frac{1}{c}\right)q(z)$

(14)

for all $x, y, z \in X$. Finally replacing x by $-x$ in (12) and using oddness of q, we obtain

$q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right) = -\left(\frac{1}{a}\right)q(x) + \left(\frac{1}{b}\right)q(y) + \left(\frac{1}{c}\right)q(z)$

(15)

for all $x, y, z \in X$. Adding the equations (12), (13), (14) and (15), we have

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right) + q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right) + q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right) + q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$

$$= 2\left(\frac{1}{a}\right)q(x) + 2\left(\frac{1}{b}\right)q(y) + 2\left(\frac{1}{c}\right)q(z)$$

(16)

for all $x, y, z \in X$. Using oddness of q in (16) and remodifying, we arrive

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$=\frac{1}{a}[q(x)-q(-x)]+\frac{1}{b}[q(y)-q(-y)]+\frac{1}{c}[q(z)-q(-z)]$$

(17)

for all $x,y,z\in X$. Adding $2\left(\frac{1}{a}\right)^2 q(x)+2\left(\frac{1}{b}\right)^2 q(y)+2\left(\frac{1}{c}\right)^2 q(z)$ on both sides, remodifying and using oddness of q, we reach (8) as desired.

Conversely, $q:X\to Y$ satisfies the functional equation (8) using oddness of in (8), we arrive

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$=2\left(\frac{1}{a}\right)q(x)+2\left(\frac{1}{b}\right)q(y)+2\left(\frac{1}{c}\right)q(z)$$

(18)

for all $x,y,z\in X$. Replacing (x,y,z) by $(x,0,0)$, $(0,x,0)$ and $(0,0,x)$, respectively in (18), we obtain

$q\left(\frac{1}{a}x\right)=\left(\frac{1}{a}\right)q(x)$, $q\left(\frac{1}{b}x\right)=\left(\frac{1}{b}\right)q(x)$ and $q\left(\frac{1}{c}x\right)=\left(\frac{1}{c}\right)q(x)$.

(19)

for all $x\in X$. One can easy to verify from (19) that

$$q\left(\frac{x}{\left(\frac{1}{a}\right)^i}\right)=\frac{1}{\left(\frac{1}{a}\right)^i}q(x) \quad ; i=1,2,3$$

(20)

for all $x\in X$. Replacing (x,y,z) by $\left(\frac{x}{\left(\frac{1}{a}\right)},\frac{y}{\left(\frac{1}{b}\right)^2},0\right)$ in equation (18) and using oddness of h and (20), we arrive our result.

3. General Solution of the Functional Equation (6): When q is Even

In this section, the general solution of the functional equation (6) for even case is given. Throughout this section, let us consider X and Y to be real vector spaces.

Theorem 3.1. If an even mapping $q:X\to Y$ satisfies the functional equation

$$q(x+y)+q(x-y)=2q(x)+2q(y)$$

(21)

for all $x, y \in X$ if and only if $q: X \to Y$ satisfies the functional equation

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$

$$=\frac{1}{a}[q(x)-q(-x)]+\frac{1}{b}[q(y)-q(-y)]+\frac{1}{c}[q(z)-q(-z)]$$

$$+2\left(\frac{1}{a}\right)^2[q(x)+q(-x)]+2\left(\frac{1}{b}\right)^2[q(y)+q(-y)]+2\left(\frac{1}{c}\right)^2[q(z)+q(-z)]$$

(22)

for all $x, y, z \in X$.

Proof. Let $q: X \to Y$ satisfies the functional equation (21). Setting (x, y) by $(0,0)$ in (21), we get $q(0)=0$. Replacing y by x and y by $2x$ in (21), we obtain

$q(2x)=4q(x)$ and $q(3x)=9q(x)$

(23)

for all $x \in X$. In general for any positive integer b , such that

$q(bx)=b^2q(x)$

(24)

for all $x \in X$. It is easy to verify from (24) that

$q(b^2x)=b^4q(x)$ and $q(b^3x)=b^6q(x)$

(25)

for all $x \in X$. Replacing (x, y) by $\left(\frac{1}{a}x,\frac{1}{b}y\right)$ in (21) and using (21), we get

$$q\left(\frac{1}{a}x+\frac{1}{b}y\right)+q\left(\frac{1}{a}x-\frac{1}{b}y\right)=2\left(\frac{1}{a}\right)^2 q(x)+2\left(\frac{1}{a}\right)^2 q(y)$$

(26)

for all $x, y \in X$. Setting (x, y) by $\left(\frac{1}{a}x,\frac{1}{c}z\right)$ in (21) and using (21), we obtain

$$q\left(\frac{1}{a}x+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{c}z\right)=2\left(\frac{1}{a}\right)^2 q(x)+2\left(\frac{1}{c}\right)^2 q(z)$$

(27)

for all $x, z \in X$. Replacing (x, y) by $\left(\frac{1}{b}y,\frac{1}{c}z\right)$ in (21) and using (21), we have

$$q\left(\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{b}y-\frac{1}{c}z\right)=2\left(\frac{1}{b}\right)^2 q(y)+2\left(\frac{1}{c}\right)^2 q(z)$$

(28)

for all $y, z \in X$. Adding equations (26), (27) and (28), we arrive

$$q\left(\frac{1}{a}x+\frac{1}{b}y\right)+q\left(\frac{1}{a}x-\frac{1}{b}y\right)+q\left(\frac{1}{a}x+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{c}z\right)+q\left(\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{b}y-\frac{1}{c}z\right)$$

$$=4\left(\frac{1}{a}\right)^2 q(x)+4\left(\frac{1}{b}\right)^2 q(y)+4\left(\frac{1}{c}\right)^2 q(z)$$

(29)

for all $x, y, z \in X$. Replacing (x, y) by $\left(\frac{1}{a}x+\frac{1}{b}y, \frac{1}{c}z\right)$ in (21), we get

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)=2q\left(\frac{1}{a}x+\frac{1}{b}y\right)+2q\left(\frac{1}{c}z\right)$$

(30)

for all $x, y, z \in X$. Setting (x, y) by $\left(\frac{1}{c}z, \frac{1}{a}x-\frac{1}{b}y\right)$ in (21), we obtain

$$q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)=2q\left(\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{b}y\right)$$

(31)

for all $x, y, z \in X$. Replacing (x, y) by $\left(\frac{1}{b}y, \frac{1}{a}x+\frac{1}{c}z\right)$ in (21), we get

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)=2q\left(\frac{1}{b}y\right)+2q\left(\frac{1}{a}x+\frac{1}{c}z\right)$$

(32)

for all $x, y, z \in X$. Setting (x, y) by $\left(\frac{1}{b}y, \frac{1}{a}x-\frac{1}{c}z\right)$ in (21), we obtain

$$q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)=2q\left(\frac{1}{b}y\right)+2q\left(\frac{1}{a}x-\frac{1}{c}z\right)$$

(33)

for all $x, y, z \in X$. Replacing (x, y) by $\left(\frac{1}{a}x, \frac{1}{b}y+\frac{1}{c}z\right)$ in (21), we get

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y-\frac{1}{c}z\right)=2q\left(\frac{1}{a}x\right)+2q\left(\frac{1}{b}y+\frac{1}{c}z\right)$$

(34)

for all $x, y, z \in X$. Setting (x, y) by $\left(\frac{1}{a}x, \frac{1}{b}y-\frac{1}{c}z\right)$ in (21), we have

$$q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)=2q\left(\frac{1}{a}x\right)+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)$$

(35)

for all $x, y, z \in X$. Now multiply by 2 on both sides of (29), we obtain

$$2q\left(\frac{1}{a}x+\frac{1}{b}y\right)+2q\left(\frac{1}{a}x-\frac{1}{b}y\right)+2q\left(\frac{1}{a}x+\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{c}z\right)+2q\left(\frac{1}{b}y+\frac{1}{c}z\right)+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)$$
$$=8\left(\frac{1}{a}\right)^2 q(x)+8\left(\frac{1}{b}\right)^2 q(y)+8\left(\frac{1}{c}\right)^2 q(z)$$

(36)

for all $x, y, z \in X$. Adding $q\left(\frac{1}{c}z\right)$ on both sides of (36), we get

$$2q\left(\frac{1}{a}x+\frac{1}{b}y\right)+2q\left(\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{b}y\right)+2q\left(\frac{1}{a}x-\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{c}z\right)+2q\left(\frac{1}{b}y+\frac{1}{c}z\right)$$
$$+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)=8\left(\frac{1}{a}\right)^2 q(x)+8\left(\frac{1}{b}\right)^2 q(y)+8\left(\frac{1}{c}\right)^2 q(z)+2q\left(\frac{1}{c}z\right)$$

(37)

for all $x, y, z \in X$. Using (30), (35) in (37), we arrive

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{c}z\right)+2q\left(\frac{1}{a}x+\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{c}z\right)$$
$$+2q\left(\frac{1}{b}y+\frac{1}{c}z\right)+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)=8\left(\frac{1}{a}\right)^2 q(x)+8\left(\frac{1}{b}\right)^2 q(y)+8\left(\frac{1}{c}\right)^2 q(z)+2\left(\frac{1}{c}\right)^2 q(z)$$
(38)

for all $x, y, z \in X$. Adding $2q\left(\frac{1}{c}z\right)$ on both sides of (38), we get

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+2q\left(\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{b}y\right)+2q\left(\frac{1}{a}x+\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{c}z\right)$$
$$+2q\left(\frac{1}{b}y+\frac{1}{c}z\right)+2q\left(\frac{1}{b}y-+\frac{1}{c}z\right)=8\left(\frac{1}{a}\right)^2 q(x)+8\left(\frac{1}{b}\right)^2 q(y)+10\left(\frac{1}{c}\right)^2 q(z)+2q\left(\frac{1}{c}z\right)$$
(39)

for all $x, y, z \in X$. Using (31) in (39), we arrive

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{c}z+\frac{1}{a}x-\frac{1}{b}y\right)+q\left(\frac{1}{c}z-\frac{1}{a}x+\frac{1}{b}y\right)$$
$$+2q\left(\frac{1}{a}x+\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{c}z\right)+2q\left(\frac{1}{b}y+\frac{1}{c}z\right)+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)$$
$$=8\left(\frac{1}{a}\right)^2 q(x)+8\left(\frac{1}{b}\right)^2 q(y)+12\left(\frac{1}{c}\right)^2 q(z)$$
(40)

for all $x, y, z \in X$. Adding $2q\left(\frac{1}{b}y\right)$ on both sides of (40), we get

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$+2q\left(\frac{1}{b}y\right)+2q\left(\frac{1}{a}x+\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{c}z\right)+2q\left(\frac{1}{b}y+\frac{1}{c}z\right)+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)$$
$$=8\left(\frac{1}{a}\right)^2 q(x)+8\left(\frac{1}{b}\right)^2 q(y)+2q\left(\frac{1}{b}y\right)+12\left(\frac{1}{c}\right)^2 q(z)$$
(41)

for all $x, y, z \in X$. Using (32) in (41), we arrive

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{c}z\right)+2q\left(\frac{1}{b}y+\frac{1}{c}z\right)+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)$$
$$=8\left(\frac{1}{a}\right)^2 q(x)+10\left(\frac{1}{b}\right)^2 q(y)+12\left(\frac{1}{c}\right)^2 q(z)$$
(42)

for all $x, y, z \in X$. Adding $2q\left(\frac{1}{b}y\right)$ on both sides of (42), we obtain

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$+q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+2q\left(\frac{1}{a}x-\frac{1}{c}z\right)+2q\left(\frac{1}{b}y\right)$$

$$+2q\left(\frac{1}{b}y+\frac{1}{c}z\right)+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)=8\left(\frac{1}{a}\right)^2 q(x)+10\left(\frac{1}{b}\right)^2 q(y)+2q\left(\frac{1}{b}y\right)+12\left(\frac{1}{c}\right)^2 q(z)$$

(43)

for all $x, y, z \in X$. Using (33) in (43), we arrive

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$+q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)$$
$$+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+2q\left(\frac{1}{b}y+\frac{1}{c}z\right)+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)$$
$$=8\left(\frac{1}{a}\right)^2 q(x)+12\left(\frac{1}{b}\right)^2 q(y)+12\left(\frac{1}{c}\right)^2 q(z)$$

(44)

for all $x, y, z \in X$. Adding $2q\left(\frac{1}{a}x\right)$ on both sides of (44), we get

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$+q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)$$
$$+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+2q\left(\frac{1}{a}x\right)+2q\left(\frac{1}{b}y+\frac{1}{c}z\right)+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)$$
$$=8\left(\frac{1}{a}\right)^2 q(x)+12\left(\frac{1}{b}\right)^2 q(y)+12\left(\frac{1}{c}\right)^2 q(z)+2q\left(\frac{1}{a}x\right)$$

(45)

for all $x, y, z \in X$. Using (34) in (45), we arrive

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$+q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$+q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y-\frac{1}{c}z\right)+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)$$
$$=10\left(\frac{1}{a}\right)^2 q(x)+12\left(\frac{1}{b}\right)^2 q(y)+12\left(\frac{1}{c}\right)^2 q(z)+2q\left(\frac{1}{a}x\right)$$

(46)

for all $x, y, z \in X$. Adding $2q\left(\frac{1}{a}x\right)$ on both sides of (46), we have

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$+q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$+q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y-\frac{1}{c}z\right)+2q\left(\frac{1}{a}x\right)+2q\left(\frac{1}{b}y-\frac{1}{c}z\right)$$
$$=10\left(\frac{1}{a}\right)^2 q(x)+12\left(\frac{1}{b}\right)^2 q(y)+12\left(\frac{1}{c}\right)^2 q(z)+2q\left(\frac{1}{a}x\right)$$

(47)

for all $x, y, z \in X$. Using (35) in (47), we arrive

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$+q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$+q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(-\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)$$
$$=12\left(\frac{1}{a}\right)^2 q(x)+12\left(\frac{1}{b}\right)^2 q(y)+12\left(\frac{1}{c}\right)^2 q(z)$$

(48)

for all $x, y, z \in X$. Using evenness of q in (48), we have

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$=4\left(\frac{1}{a}\right)^2 q(x)+4\left(\frac{1}{b}\right)^2 q(y)+4\left(\frac{1}{c}\right)^2 q(z)$$

(49)

for all $x, y, z \in X$. Using evenness of q in (49) one can get,

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$=2\left(\frac{1}{a}\right)^2[q(x)+q(-x)]+2\left(\frac{1}{b}\right)^2[q(y)+q(-y)]+2\left(\frac{1}{c}\right)^2[q(z)+q(-z)]$$

(50)

for all $x, y, z \in X$. Adding $\frac{1}{a}q(x)+\frac{1}{b}q(y)+\frac{1}{c}q(z)$ on both sides of (50) and using evenness of q, we desired our result.

Conversely, $q: X \to Y$ satisfies the functional equation (22). Using evenness of q in (22), we have

$$q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right)+q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right)$$
$$=4\left(\frac{1}{a}\right)^2 q(x)+4\left(\frac{1}{b}\right)^2 q(y)+4\left(\frac{1}{c}\right)^2 q(z)$$

(51)

Setting (x, y, z) by $(x,0,0)$, $(0,x,0)$ and $(0,0,x)$ in (51), we obtain

$$q\left(\frac{1}{a}x\right)=\left(\frac{1}{a}\right)^2 q(x); \quad q\left(\frac{1}{b}y\right)=\left(\frac{1}{b}\right)^2 q(x) \quad \text{and} \quad q\left(\frac{1}{c}z\right)=\left(\frac{1}{c}\right)^2 q(x)$$

(52)

for all $x \in X$. It is easy to verify from (52), that

$$q\left(\frac{x}{\left(\frac{1}{a}\right)^i}\right)=\frac{1}{\left(\frac{1}{a}\right)^i}q(x), \quad i=1,2,3$$

(53)

for all $x \in X$. Replacing (x,y,z) by $\left(\frac{x}{\left(\frac{1}{a}\right)}, \frac{y}{\left(\frac{1}{b}\right)}, 0\right)$ in (51) and using evenness of q and (53), we desired our result.

For sections 4, 5 and 6, let us consider X and Y to a normed space and a Banach space. Define a mapping $Dq : X \to Y$ by

$$Dq(x,y,z) = q\left(\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right) + q\left(\frac{1}{a}x-\frac{1}{b}y+\frac{1}{c}z\right) + q\left(\frac{1}{a}x+\frac{1}{b}y-\frac{1}{c}z\right)$$

$$+q\left(-\frac{1}{a}x+\frac{1}{b}y+\frac{1}{c}z\right) - \frac{1}{a}[q(x)-q(-x)] - \frac{1}{b}[q(y)-q(-y)] - \frac{1}{c}[q(z)-q(-z)]$$

$$-2\left(\frac{1}{a}\right)^2[q(x)+q(-x)] - 2\left(\frac{1}{b}\right)^2[q(y)+q(-y)] - 2\left(\frac{1}{c}\right)^2[q(z)+q(-z)]$$

for all $x, y, z \in X$.

Theorem 2.2. Let $j \in \{-1,1\}$ and $\alpha : X^3 \to [0,\infty)$ be a function such that

$$\sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{1}{a}\right)^{kj} x, \left(\frac{1}{a}\right)^{kj} y, \left(\frac{1}{a}\right)^{kj} z\right)}{\left(\frac{1}{a}\right)^{kj}}$$ converges in $\mathbb{R}$ and $$\lim_{k\to\infty} \frac{\alpha\left(\left(\frac{1}{a}\right)^{kj} x, \left(\frac{1}{a}\right)^{kj} y, \left(\frac{1}{a}\right)^{kj} z\right)}{\left(\frac{1}{a}\right)^{kj}} = 0$$

(54)

for all $x, y, z \in X$. Let $q_a : X \to Y$ be an odd function satisfying the inequality

$$\| Dq_a(x,y,z) \| \le \alpha(x,y,z)$$

(55)

for all $x, y, z \in X$. There exists a unique additive mapping $A : X \to Y$ which satisfies the functional equation (6) and

$$\| q_a(x) - A(x) \| \le \frac{1}{2\left(\frac{1}{a}\right)} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(\left(\frac{1}{a}\right)^{kj} x, 0, 0\right)}{\left(\frac{1}{a}\right)^{kj}}$$

(56)

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} \frac{q_a\left(\left(\frac{1}{a}\right)^{kj} x\right)}{\left(\frac{1}{a}\right)^{kj}}$$

(57)

for all $x \in X$.

Proof. Assume that $j = 1$. Replacing (x,y,z) and $(x,0,0)$ in (55) and using oddness of q_a, we get

$$\left\| 2g_a\left(\frac{1}{a}x\right) - 2\left(\frac{1}{a}\right)g_a(x) \right\| \leq \alpha(x,0,0)$$

(58)

for all $x \in X$. It follows from (58) that

$$\left\| \frac{q_a\left(\frac{1}{a}x\right)}{\left(\frac{1}{a}\right)} - q_a(x) \right\| \leq \frac{\alpha}{2\left(\frac{1}{a}\right)}(x,0,0)$$

(59)

for all $x \in X$. Replacing x by $\left(\frac{1}{a}\right)x$ in (59) and dividing by $\left(\frac{1}{a}\right)$, we obtain

$$\left\| \frac{q_a\left(\frac{1}{b}x\right)}{\left(\frac{1}{b}\right)} - \frac{q_a\left(\frac{1}{a}x\right)}{\left(\frac{1}{a}\right)} \right\| \leq \frac{\alpha}{2\left(\frac{1}{b}\right)}\left(\left(\frac{1}{a}\right)x,0,0\right)$$

(60)

for all $x \in X$. It follows from (59) and (60) that

$$\left\| \frac{q_a\left(\frac{1}{b}x\right)}{\left(\frac{1}{b}\right)} - q_a(x) \right\| \leq \frac{1}{2\left(\frac{1}{a}\right)}\left[\alpha(x,0,0) + \frac{\alpha}{\left(\frac{1}{a}\right)}\left(\left(\frac{1}{a}\right)x,0,0\right)\right]$$

(61)

for all $x \in X$. Generalizing, we have

$$\left\| q_a(x) - \frac{q_a\left(\left(\frac{1}{a}\right)^k x\right)}{\left(\frac{1}{a}\right)^k} \right\| \leq \frac{1}{2\left(\frac{1}{a}\right)} \sum_{k=0}^{n-1} \frac{\alpha\left(\left(\frac{1}{a}\right)^k x,0,0\right)}{\left(\frac{1}{a}\right)^k} \leq \frac{1}{2\left(\frac{1}{a}\right)} \sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{1}{a}\right)^k x,0,0\right)}{\left(\frac{1}{a}\right)^k}$$

(62)

for all $x \in X$. In order to prove convergence of the sequence

$$\left\{ \frac{q_a\left(\frac{1}{a}\right)^k x}{\left(\frac{1}{a}\right)^k} \right\},$$

Replace x by $\left(\frac{1}{a}\right)^l x$ and dividing $\left(\frac{1}{a}\right)^l$ (62), for any $k,l > 0$, to deduce

$$\left\| \frac{q_a\left(\left(\frac{1}{a}\right)^l x\right)}{\left(\frac{1}{a}\right)^l} - \frac{q_a\left(\left(\frac{1}{a}\right)^{k+l} x\right)}{\left(\frac{1}{a}\right)^{k+l}} \right\| = \frac{1}{2^l}\left\| q_a\left(\left(\frac{1}{a}\right)^l x\right) - \frac{q_a\left(\left(\frac{1}{a}\right)^k . \left(\frac{1}{a}\right)^l x\right)}{\left(\frac{1}{a}\right)^k} \right\|$$

$$\le \frac{1}{2\left(\frac{1}{a}\right)} \sum_{k=0}^{n-1} \frac{\alpha\left(\left(\frac{1}{a}\right)^{k+l} x, 0, 0\right)}{\left(\frac{1}{a}\right)^{k+l}}$$

(63)

$$\le \frac{1}{2\left(\frac{1}{a}\right)} \sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{1}{a}\right)^{k+l} x, 0, 0\right)}{\left(\frac{1}{a}\right)^{k+l}} \to 0 \text{ as } l \to \infty$$

for all $x \in X$.

Hence the sequence $\left\{ \frac{g_a\left(\left(\frac{1}{a}\right)^k x\right)}{\left(\frac{1}{a}\right)^k} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A: X \to Y$ such that

$$A(x) = \lim_{k\to\infty} \frac{q_a\left(\left(\frac{1}{a}\right)^k x\right)}{\left(\frac{1}{a}\right)^k}, \qquad \forall\, x \in X$$

.

Letting $k \to \infty$ in (62), we see that (56) holds for all $x \in X$. To prove that A satisfies (6), replacing (x, y, z) by $\left(\left(\frac{1}{a}\right)^k x, \left(\frac{1}{a}\right)^k y, \left(\frac{1}{a}\right)^k z\right)$ and dividing $\left(\frac{1}{a}\right)^k$ in (54), we obtain

$$\frac{1}{\left(\frac{1}{a}\right)^k} \left\| Dq_a\left(\left(\frac{1}{a}\right)^k x, \left(\frac{1}{a}\right)^k y, \left(\frac{1}{a}\right)^k z\right) \right\| \le \frac{1}{\left(\frac{1}{a}\right)^k} \alpha\left(\left(\frac{1}{a}\right)^k x, \left(\frac{1}{a}\right)^k y, \left(\frac{1}{a}\right)^k z\right)$$

for all $x, y, z \in X$. Letting $k \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$DA(x, y, z) = 0$$

.

Hence A satisfies (6) for all $x, y, z \in X$. To show that A is unique, let $B(x)$ be another additive mapping satisfying (6) and (56), then

$$\| A(x) - B(x) \|$$

$$= \frac{1}{\left(\frac{1}{a}\right)^l} \left\| A\left(\left(\frac{1}{a}\right)^l x\right) - B\left(\left(\frac{1}{a}\right)^l x\right) \right\|$$

$$\le \frac{1}{\left(\frac{1}{a}\right)^l} \left\| A\left(\left(\frac{1}{a}\right)^l x\right) - q_a\left(\left(\frac{1}{a}\right)^l x\right) \right\| + \left\| q_a\left(\left(\frac{1}{a}\right)^l x\right) - B\left(\left(\frac{1}{a}\right)^l x\right) \right\|$$

$$\leq \frac{1}{2\left(\frac{1}{a}\right)} \sum_{k=0}^{\infty} \frac{\alpha\left(\left(\frac{1}{a}\right)^{k+l} x, 0, 0\right)}{\left(\frac{1}{a}\right)^{k+l}} \to 0 \text{ as } l \to \infty$$

for all $x \in X$. Hence A is unique.

Now, replacing x by $\frac{x}{\left(\frac{1}{a}\right)}$ in (58), we get

$$\left\| 2q_a(x) - 2\left(\frac{1}{a}\right) q_a\left(\frac{x}{\left(\frac{1}{a}\right)}\right) \right\| \leq \alpha\left(\frac{x}{\left(\frac{1}{a}\right)}, 0, 0\right)$$

(64)

for all $x \in X$. It follows from (64) that

$$\left\| q_a(x) - \left(\frac{1}{a}\right) q_a\left(\frac{x}{\left(\frac{1}{a}\right)}\right) \right\| \leq \frac{1}{2}\alpha\left(\frac{x}{\left(\frac{1}{a}\right)}, 0, 0\right)$$

(65)

for all $x \in X$. The rest of the proof is similar to that of $j = 1$. Hence for $j = -1$ also the theorem is true. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 4.1 concerning the stability of (6).

Corollary 2.3. Let λ and s be a nonnegative real numbers. Let an odd function $q_a : X \to Y$ satisfying the inequality

$$\| Dq_a(x, y, z) \| \leq \begin{cases} \lambda \\ \lambda\left(\|x\|^s + \|y\|^s + \|z\|^s\right), & s \neq 1; \\ \lambda\left(\|x\|^s \|y\|^s \|z\|^s + \left\{\|x\|^{3s} + \|y\|^{3s} + \|z\|^{3s}\right\}\right), & s \neq \frac{1}{3}; \end{cases}$$

(66)

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\| q_a(x) - A(x) \| \leq \begin{cases} \dfrac{\lambda}{2\left|\left(\frac{1}{a}\right)-1\right|}, \\ \dfrac{\lambda\|x\|^s}{2\left|\left(\frac{1}{a}\right)-\left(\frac{1}{a}\right)^s\right|}, \\ \dfrac{\lambda\|x\|^{3s}}{2\left|\left(\frac{1}{a}\right)-\left(\frac{1}{c}\right)^s\right|}, \end{cases}$$

(67)

for all $x \in X$.

Proof: If we replace

$$\alpha(x,y,z) = \begin{cases} \lambda; \\ \lambda\left(\|x\|^s + \|y\|^s + \|z\|^s\right); \\ \lambda\left(\|x\|^s\|y\|^s\|z\|^s + \left\{\|x\|^{3s} + \|y\|^{3s} + \|z\|^{3s}\right\}\right); \end{cases}$$

(68)

in Theorem 4.1, we arrive (67).

From the above results, we prove the even and mixed case methods.

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