Improved and Generalized Bernstein Type Inequalities for the Higher Derivatives of a Polynomial

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Abstract- Let $p(z)$ be a polynomial of degree n having no zero zero in $|z|<k$, $k\ge 1$, then for $1\le R\le k$, Dewan and Bidkham [J. Math. Anal. Appl., 166(1992), 319-324] proved

$$\max_{|z|=R}|p'(z)|\le n\frac{(R+k)^{n-1}}{(1+k)^{n}}\max_{|z|=1}|p(z)|.$$

The result is best possible and extremal polynomial is $p(z)=(z+k)^n$.

In this paper, by involving certain coefficients of the polynomial $p(z)$, we prove a result concerning the estimate of maximum modulus of higher derivatives of $p(z)$, which not only improves as well as generalizes the above result, but also has interesting consequences as special cases.

Keywords – Polynomial, Polynomial Inequalities, Maximum Modulus.

I. Introduction

It was for the first time, Bernstein [12, 13] investigated an upper bound for the maximum modulus of the first derivative of a complex polynomial on the unit circle in terms the maximum modulus of the polynomial on the same circle and proved the following famous result known as Bernstein's inequality that if $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|p'(z)|\le n\max_{|z|=1}|p(z)|. \tag{1.1}$$

Inequality (1.1) is best possible and equality occurs for $p(z)=\lambda z^n$, $\lambda\ne 0$, is any complex number.

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If we restrict to the class of polynomials having no zero in $|z|<1$, then inequality (1.1) can be sharpened as

$$\max_{|z|=1}\left|p'(z)\right| \le \frac{n}{2}\max_{|z|=1}\left|p(z)\right|. \tag{1.2}$$

The result is sharp and equality holds in (1.2) for $p(z)=\alpha+\beta z^n$, where $|\alpha|=|\beta|$.
Inequality (1.2) was conjectured by Erdös and later proved by Lax [10].
Simple proofs of this theorem were later given by de-Bruijn [3], and Aziz and Mohammad [1].
It was asked by Professor R.P. Boas that if $p(z)$ is a polynomial of degree n not vanishing in $|z|<k$, $k>0$, then how large can

$$\left\{ \max_{|z|=1}\left|p'(z)\right| \Big/ \max_{|z|=1}\left|p(z)\right| \right\} \text{ be?} \tag{1.3}$$

A partial answer to this problem was given by Malik [11], who proved

Theorem A. *If $p(z)$ is a polynomial of degree n having no zero in the disc $|z|<k$, $k\ge 1$, then*

$$\max_{|z|=1}\left|p'(z)\right| \le \frac{n}{1+k}\max_{|z|=1}\left|p(z)\right| \tag{1.4}$$

The result is best possible and equality holds for $p(z)=(z+k)^n$.

For the class of polynomials not vanishing in $|z|<k$, $k\le 1$, the precise estimate for maximum of $\left|p'(z)\right|$ on $|z|=1$, in general, does not seem to be easily obtainable. For quite some time, it was believed that if $p(z)\ne 0$ in $|z|<k$, $k\le 1$, then the inequality analogous to (1.4) should be

$$\max_{|z|=1}\left|p'(z)\right| \le \frac{n}{1+k^n}\max_{|z|=1}\left|p(z)\right|. \tag{1.5}$$

till Professor E.B. Saff gave the example $p(z)=\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)$ to counter this belief.

Dewan and Bidkham [4] generalized Theorem A by considering any circle that lies in a closed circular annulus of radii 1 and k where $k\ge 1$. In fact, they prove

Theorem B. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, $k\ge 1$, then for $1\le R\le k$,*

$$\max_{|z|=R}\left|p'(z)\right| \le n\frac{(R+k)^{n-1}}{(1+k)^n}\max_{|z|=1}\left|p(z)\right|. \tag{1.6}$$

The result is best possible and extremal polynomial is $p(z)=(z+k)^n$.

In this paper, by involving some coefficients of the polynomial $p(z)$ and also $\min_{|z|=k}\left|p(z)\right|$, we obtain a result which is an improvement and a generalization of (1.6) by further extending for the sth derivative of $p(z)$ and maxima are considered on two different circles lying both inside and on any circle . More precisely, we have

Theorem. *If $p(z)=\sum_{\upsilon=0}^{n}a_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z|<k$, $k>0$, then for $0<r\le R\le k$, $1\le s\le n$, and for every real or complex number λ with $|\lambda|<1$,*

$$\max_{|z|=R}\left|p^{(s)}(z)\right| \le \frac{n(n-1)\cdots(n-s+1)}{R^s+\delta_{k,s}}\left[\left(\frac{R+k}{r+k}\right)^n \max_{|z|=r}\left|p(z)\right| - \left\{\left(\frac{R+k}{r+k}\right)^n - 1 + |\lambda|\right\}\min_{|z|=k}\left|p(z)\right|\right],$$

(1.7)

where

$$\delta_{k,s} = \frac{k^{s+1} + \dfrac{1}{C(n,s)}\dfrac{|a_s|}{|a_0| - |\lambda| m}k^{2s}R}{R + \dfrac{1}{C(n,s)}\dfrac{|a_s|}{|a_0| - |\lambda| m}k^{s+1}} \quad \textit{with} \quad C(n,s) = \frac{n!}{s!(n-s)!}.$$

The result is best possible for $s=1$ *and equality in* (1.7) *holds for* $p(z)=(z+k)^n$.

Remark 1.1. Taking limit as $|\lambda| \to 1$, our theorem reduces to the following interesting result, which gives a generalization of the result of Mir [13].

Corollary 1.1. *If* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree n having no zero in* $|z|<k$, $k>0$, *then for* $0<r\le R\le k$ *and* $1\le s\le n$,

$$\max_{|z|=R}\left|p^{(s)}(z)\right| \le \frac{n(n-1)...(n-s+1)}{R^s+\phi_{k,s}}\left(\frac{R+k}{r+k}\right)^n\left\{\max_{|z|=r}|p(z)| - \min_{|z|=k}|p(z)|\right\}, \quad (1.8)$$

where

$$\phi_{k,s} = \frac{k^{s+1} + \dfrac{1}{C(n,s)}\dfrac{|a_s|}{|a_0| - m}k^{2s}R}{R + \dfrac{1}{C(n,s)}\dfrac{|a_s|}{|a_0| - m}k^{s+1}}.$$

Remark 1.2. Putting $R = r =1$, Corollary 1.1 further reduces to the result proved by Mir [13], namly

Corollary 1.2. *If* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z|<k$, $k\ge 1$, *then for* $1\le s\le n$,

$$\max_{|z|=1}\left|p^{(s)}(z)\right| \le \frac{n(n-1)...(n-s+1)}{\psi_{k,s}+1}\left\{\max_{|z|=1}|p(z)| - \min_{|z|=k}|p(z)|\right\},$$

where

$$\psi_{k,s} = k^{s+1}\left\{\frac{1+\dfrac{1}{C(n,s)}\dfrac{|a_s|}{|a_0|-m}k^{s-1}}{1+\dfrac{1}{C(n,s)}\dfrac{|a_s|}{|a_0|-m}k^{s+1}}\right\}.$$

Remark 1.3. Further in Corollary 1.2, it is seen similar in the proof of the theorem that for a complex number λ with $|\lambda|<1$, polynomial $p(z)-\lambda m$, where $m=\min_{|z|=k}|p(z)|$, has no zero in $|z|<k$, $k\ge 1$ and it follows by applying inequality (2.2) of Lemma 2.1 to $p(Rz)-\lambda m$, that

$$C(n,s)\left|a_0 - \lambda m\right| \geq \left|a_s\right| k^s$$

By Lemma 2.3, we get

$$C(n,s)\left(\left|a_0\right| - \left|\lambda\right| m\right) \geq \left|a_s\right| k^s .$$

Letting $|\lambda| \to 1$, we get

$$C(n,s)\left(\left|a_0\right| - m\right) \geq \left|a_s\right| k^s ,$$

which in turn implies $\psi_{k,s}$ of Corollary 1.2 is such that $\psi_{k,s} \geq k^s$ for $1 \leq s \leq n$. This implies that Corollary 1.2 is an improvement of a result of Govil [8].

Remark 1.4. Putting $\lambda = 0$, our theorem reduces to the following result, which is a generalization of a result of Aziz and Rather [2].

Corollary 1.3. *If* $p(z) = \sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree n having no zero in* $|z| < k$, $k > 0$ *, then for* $0 < r \leq R \leq k$ *, and* $1 \leq s \leq n$ *,*

$$\max_{|z|=R}\left|p^{(s)}(z)\right| \leq \frac{n(n-1)\cdots(n-s+1)}{R^s + \alpha_{k,s}}\left[\left(\frac{R+k}{r+k}\right)^n \max_{|z|=r}\left|p(z)\right| - \left\{\left(\frac{R+k}{r+k}\right)^n - 1\right\}\min_{|z|=k}\left|p(z)\right|\right]. \quad (1.8)$$

where

$$\alpha_{k,s} = \frac{k^{s+1} + \frac{1}{C(n,s)}\frac{|a_s|}{|a_0|}k^{2s}R}{R + \frac{1}{C(n,s)}\frac{|a_s|}{|a_0|}k^{s+1}}.$$

Remark 1.5. If we put $R = r = 1$ in Corollary 1.1 then in this case $k \geq 1$ and by Lemma 2.1, we have

$$(k-1)\left\{1 - \frac{1}{C(n,s)}\frac{|a_s|}{|a_0|}k^s\right\} \geq 0 ,$$

which is equivalent to

$$\frac{k^{s+1} + \frac{1}{C(n,s)}\frac{|a_s|}{|a_0|}k^{2s}}{1 + \frac{1}{C(n,s)}\frac{|a_s|}{|a_0|}k^{s+1}} \geq k^s . \quad (1.9)$$

Inequality (1.9) clearly shows that Corollary 1.1 is an improvement and a generalization of a result proved by Govil and Rahman [7].

Remark 1.5. Corollary 1.1 provides a generalization of a result due Aziz and Rather [2].

Remark 1.5. If we assign $\lambda = 0$ and $R = r = s = 1$, our theorem reduces to a result of Govil et. al [8].

Remark 1.6. Corollary 1.2 is an improved as well as a generalised version of a well-known inequality proved by Malik [11] under the same set of hypotheses.

Remark 1.7. If we put $s = r = R = k = 1$, Corollary 1.1 gives an improvement of inequality (1.2), conjectured by Erdös and later proved by Lax [10].

II LEMMA

The following lemmas are needed for the proof of the theorem.

Lemma 2.1. *If* $p(z)=\sum_{\upsilon=0}^{n} a_{\upsilon} z^{\upsilon}$ *is a polynomial of degree* n *such that* $p(z)\neq 0$ *for* $|z|<k$, $k\geq 1$, *then*

$$\mu_{k,s}\left|p^{(s)}(z)\right|\leq\left|q^{(s)}(z)\right| \text{ for } |z|=1, \tag{2.1}$$

and

$$\frac{1}{C(n,s)}\frac{|a_s|}{|a_0|}k^{s}\leq 1, \tag{2.2}$$

where

$$q(z)=z^{n}\,\overline{p\left(\frac{1}{\bar{z}}\right)}, \quad \mu_{k,s}=\frac{C(n,s)|a_0|k^{s+1}+|a_s|k^{2s}}{C(n,s)|a_0|+|a_s|k^{s+1}}$$ and $C(n,s)$ *is as defined in our theorem.*

The above result is due to Aziz and Rather [2].

Lemma 2.2. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n} a_{\upsilon} z^{\upsilon}$, $1\leq\mu\leq n$, *is a polynomial of degree n such that* $p(z)\neq 0$ *in* $|z|<k$, $k>0$, *then for* $0<r\leq R\leq k$,

$$\max_{|z|=R}|p(z)|\leq\left(\frac{R^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}\max_{|z|=r}|p(z)|-\left\{\left(\frac{R^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}-1\right\}\min_{|z|=k}|p(z)|, \tag{2.3}$$

Equality holds in (2.2) *for* $p(z)=\left(z^{\mu}+k^{\mu}\right)^{\frac{n}{\mu}}$ *where n is a multiple of* μ.

This Lemma is due to Dewan et. al [5].

Lemma 2.3. *If* $p(z)=\sum_{\upsilon=0}^{n} a_{\upsilon} z^{\upsilon}$ *is a polynomial of degree n with* $p(z)\neq 0$ *for* $|z|<k$, $k\geq 1$, *then* $|p(z)|>m$ *for* $|z|<k$, *and in particular* $|a_0|>m$, *where* $m=\min_{|z|=k}|p(z)|$.

The above lemma is due to Gardner et al. [6].

The following lemma was proved by Govil [7].

Lemma 2.4. *If* $p(z)$ *is a polynomial of degree n having no zero in* $|z|<k$, $k\geq 1$, *then for* $|z|>\frac{1}{k}$,

$$\left|q^{(s)}(z)\right|\geq mn(n-1)\ldots(n-s+1)|z|^{n-s}, \tag{2.4}$$

where $m=\min_{|z|=k}|p(z)|$ and $q(z)=z^{n}\,\overline{p\left(\frac{1}{\bar{z}}\right)}$.

Lemma 2.5. *The function*

$$\phi(x) = k^{s+1}\left\{\frac{1+\dfrac{1}{C(n,s)}\left(\dfrac{|a_s|}{x}\right)k^{s-1}}{1+\dfrac{1}{C(n,s)}\left(\dfrac{|a_s|}{x}\right)k^{s+1}}\right\}, k \geq 1$$

is an increasing function of x .

$C(n, s)$ is as defined in our theorem.

Proof of Lemma 2.5. The proof follows easily by considering the first derivative test on $\phi(x)$.

III. PROOF OF THE THEOREM

If $p(z)$ has no zero in $|z|<k$, $k>0$ and if $0<r\leq R\leq k$, then $P(z)=p(Rz)$ has no zero in $|z|<\frac{k}{R}$, $\frac{k}{R}\geq 1$. Now, $m=\min_{|z|=\frac{k}{R}}|P(z)|=\min_{|z|=\frac{k}{R}}|p(Rz)|=\min_{|z|=k}|p(z)|$.

If $m>0$, then it follows from Rouche's Theorem that for every real or complex number λ with $|\lambda|<1$ the polynomial $F(z)=P(z)-\lambda m$ has no zero in $|z|<\frac{k}{R}$, $\frac{k}{R}\geq 1$. Applying inequality (2.1) of Lemma 2.1 to $F(z)$, we have for $|z|=1$

$$\mu'_{k,s}\left|P^{(s)}(z)\right| \leq \left|Q^{(s)}(z)-\bar{\lambda}mn(n-1)...(n-s+1)z^{n-s}\right| , \tag{3.1}$$

where

$$Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)},$$

$$\mu'_{k,s}=\frac{1}{R^s}\left\{\frac{k^{s+1}+\dfrac{1}{C(n,s)}\dfrac{|a_s|}{|a_0-\lambda m|}k^{2s}R}{R+\dfrac{1}{C(n,s)}\dfrac{|a_s|}{|a_0-\lambda m|}k^{s+1}}\right\},$$

and

$$C(n,s)=\frac{n!}{s!(n-s)!}.$$

Since $|\lambda|<1$, we have

$$|a_0-\lambda m|\geq\big||a_0|-|\lambda|m\big|=|a_0|-|\lambda|m. \tag{3.2}$$

In view of inequality (3.2) and Lemma 2.5, inequality (3.1) gives for $|z|=1$,

$$\mu_{k,s}\left|P^{(s)}(z)\right| \leq \left|Q^{(s)}(z)-\bar{\lambda}mn(n-1)...(n-s+1)z^{n-s}\right| , \tag{3.3}$$

where

$$\mu_{k,s}=\frac{1}{R^s}\left\{\frac{k^{s+1}+\dfrac{1}{C(n,s)}\dfrac{|a_s|}{|a_0|-|\lambda|m}k^{2s}R}{R+\dfrac{1}{C(n,s)}\dfrac{|a_s|}{|a_0|-|\lambda|m}k^{s+1}}\right\}.$$

If we choose the argument of λ on the right hand side of (3.3) such that for $|z|=1$,

$$\left|Q^{(s)}(z)-\bar{\lambda}mn(n-1)...(n-s+1)z^{n-s}\right|=\left|Q^{(s)}(z)\right|-|\lambda|mn(n-1)...(n-s+1), \tag{3.4}$$

which is possible by inequality (2.3) of Lemma 2.4.
Using (3.4) to (3.3), we obtain

$$\mu_{k,s}\left|P^{(s)}(z)\right| \le \left|Q^{(s)}(z)\right| - |\lambda| mn(n-1)\ldots(n-s+1), \text{ for } |z| = 1. \quad (3.5)$$

Now, if $f(z)$ is a polynomial of degree n having all its zeros in $|z| \le 1$, then $g(z) = z^n \overline{f\left(\frac{1}{\bar{z}}\right)}$ has no zero in $|z| < 1$. Hence, using inequality (2.1) of Lemma 2.1 with $k = 1$, we have for $|z| = 1$

$$|g^{(s)}(z)| \le |f^{(s)}(z)|. \quad (3.6)$$

Let $M = \max_{|z|=1} |P(z)|$, then for every real or complex number γ with $|\gamma| > 1$ we have, by Rouche's theorem the polynomial $T(z) = P(z) - \gamma M z^n$ has all its zeros in $|z| < 1$. Suppose

$$S(z) = z^n \overline{T\left(\frac{1}{\bar{z}}\right)} = z^n \overline{P\left(\frac{1}{\bar{z}}\right)} - \bar{\gamma} M = Q(z) - \bar{\gamma} M$$

Applying (3.6) to $T(z)$, we get for $1 \le s \le n$ and $|z| = 1$

$$\left|S^{(s)}(z)\right| \le \left|T^{(s)}(z)\right|,$$

which implies

$$\left|Q^{(s)}(z)\right| \le \left|P^{(s)}(z) - \gamma M n(n-1)\ldots(n-s+1)z^{n-s}\right| \text{ for } |z| = 1. \quad (3.7)$$

Since $P(z)$ is of degree n, it is evident that the polynomial $P^{(s)}(z)$ is of degree $(n - s)$ and repeated application of Bernstein's inequality (1.1) to $P(z)$ yields for $|z| = 1$

$$\left|P^{(s)}(z)\right| \le M n(n-1)\ldots(n-s+1). \quad (3.8)$$

Further, for suitable choice of the argument of γ in (3.7), we have for $|z| = 1$

$$\left|Q^{(s)}(z)\right| \le \left||\gamma| M n(n-1)\ldots(n-s+1) - \left|P^{(s)}(z)\right|\right|,$$

which becomes by taking limit as $|\gamma| \to 1$,

$$\left|Q^{(s)}(z)\right| \le \left|M n(n-1)\ldots(n-s+1) - \left|P^{(s)}(z)\right|\right| = M n(n-1)\ldots(n-s+1) - \left|P^{(s)}(z)\right|,$$

which for $|z| = 1$, becomes on using (3.8) that

$$\left|P^{(s)}(z)\right| + \left|Q^{(s)}(z)\right| \le n(n-1)\ldots\ldots(n-s+1)M. \quad (3.9)$$

Inequality (3.9) in conjunction with inequality (3.5) gives for $|z| = 1$

$$\left|P^{(s)}(z)\right| \le \frac{1}{1+\mu_{k,s}} n(n-1)\ldots\ldots(n-s+1)\left\{\max_{|z|=1} |P(z)| - |\lambda| m\right\}, \quad (3.10)$$

which on replacing $P(z)$ by $p(Rz)$, we have for $|z| = 1$,

$$\left|p^{(s)}(z)\right| \le \frac{1}{R^s + \delta_{k,s}} n(n-1)\ldots\ldots(n-s+1)\left\{\max_{|z|=R} |p(z)| - |\lambda| \min_{|z|=k} |p(z)|\right\} \text{ for } |z| = 1, \quad (3.11)$$

where $\delta_{k,s}$ is as defined in the theorem.
When $\mu = 1$, Lemma 2.2 becomes for $0 < r \le R \le k$,

$$\max_{|z|=R} |p(z)| \le \left(\frac{R+k}{r+k}\right)^n \max_{|z|=r} |p(z)| - \left(\left(\frac{R+k}{r+k}\right)^n - 1\right) \min_{|z|=k} |p(z)|. \quad (3.12)$$

Using inequality (3.12) to inequality (3.11), we obtain

$$\max_{|z|=R}|p^{(s)}(z)| \le \frac{n(n-1)...(n-s+1)}{R^s+\delta_{k,s}}\left[\left(\frac{R+k}{r+k}\right)^n \max_{|z|=r}|p(z)| - \left\{\left(\frac{R+k}{r+k}\right)^n - 1 + \lambda\right\}\min_{|z|=k}|p(z)|\right],$$

which completes the proof of the theorem.

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