Improved and Generalized Bernstein Type Inequalities for the Higher Derivatives of a Polynomial

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Abstract - Let $p(z)$ be a polynomial of degree $n$ having no zero zero in $z \leq k$, $k \geq 1$, then for $1 \leq R \leq k$, Dewan and Bidkham [J. Math. Anal. Appl., 166(1992), 319-324] proved
\[
\max_{|z|=R}|p(z)| \leq n \left( \frac{R+k}{1+k} \right)^{n-1} \max_{|z|=1}|p(z)|.
\]
The result is best possible and extremal polynomial is $p(z) = (z+k)^n$.

In this paper, by involving certain coefficients of the polynomial $p(z)$, we prove a result concerning the estimate of maximum modulus of higher derivatives of $p(z)$, which not only improves as well as generalizes the above result, but also has interesting consequences as special cases.

Keywords – Polynomial, Polynomial Inequalities, Maximum Modulus.

I. INTRODUCTION

It was for the first time, Bernstein [12, 13] investigated an upper bound for the maximum modulus of the first derivative of a complex polynomial on the unit circle in terms the maximum modulus of the polynomial on the same circle and proved the following famous result known as Bernstein’s inequality that if $p(z)$ is a polynomial of degree $n$, then
\[
\max_{|z|=1}|p(z)| \leq n \max_{|z|=1}|p(z)|.
\] (1.1)

Inequality (1.1) is best possible and equality occurs for $p(z) = \lambda z^n$, $\lambda \neq 0$, is any complex number.

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If we restrict to the class of polynomials having no zero in $z \leq k$, then inequality (1.1) can be sharpened as
Theorem. If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then
\[
\max_{|z|=1} |p(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|.
\] (1.2)

The result is sharp and equality holds in (1.2) for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$

Inequality (1.2) was conjectured by Erdős and later proved by Lax [10].

Simple proofs of this theorem were later given by de-Bruijn [3], and Aziz and Mohammad [1].

It was asked by Professor R.P. Boas that if $p(z)$ is a polynomial of degree $n$ not vanishing in $|z| < k$, $k > 0$, then how large can
\[
\max_{|z|=1} \left| \frac{p(z)}{\max_{|z|=1} |p(z)|} \right| \text{ be?}
\] (1.3)

A partial answer to this problem was given by Malik [11], who proved

**Theorem A.** If $p(z)$ is a polynomial of degree $n$ having no zero in the disc $|z| < k$, $k \geq 1$, then
\[
\max_{|z|=1} |p(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|.
\] (1.4)

The result is best possible and equality holds for $p(z) = (z+k)^n$.

For the class of polynomials not vanishing in $|z| < k$, $k \leq 1$, the precise estimate for maximum of $|p(z)|$ on $|z| = 1$, in general, does not seem to be easily obtainable. For quite some time, it was believed that if $p(z) \neq 0$ on $|z| < k$, $k \leq 1$, then the inequality analogous to (1.4) should be
\[
\max_{|z|=1} |p(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|.
\] (1.5)

till Professor E.B. Saff gave the example $p(z) = \frac{z - \frac{1}{2}}{z + 3}$ to counter this belief.

Dewan and Bidkham [4] generalized Theorem A by considering any circle that lies in a closed circular annulus of radii 1 and $k$ where $k \geq 1$. In fact, they prove

**Theorem B.** If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then for $1 \leq R \leq k$,
\[
\max_{|z|=R} |p(z)| \leq \frac{n}{(1+k)^n} \max_{|z|=1} |p(z)|.
\] (1.6)

The result is best possible and extremal polynomial is $p(z) = (z+k)^n$.

In this paper, by involving some coefficients of the polynomial $p(z)$ and also $\min_{|z|=k} |p(z)|$, we obtain a result which is an improvement and a generalization of (1.6) by further extending for the $s$th derivative of $p(z)$ and maxima are considered on two different circles lying both inside and on any circle. More precisely, we have

**Theorem.** If $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^\nu$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$, $1 \leq s \leq n$, and for every real or complex number $\lambda$ with $\lambda \neq 1$,
\[
\max_{|z|=R} |p^{(s)}(z)| \leq \frac{n(n-1)\ldots(n-s+1)}{R^n + \delta_{k,s}} \left[ \left| \frac{R+k}{r+k} \right| \max_{|z|=R} |p(z)| - \min_{|z|=R} |p(z)| \right],
\]
(1.7)
where
\[
\delta_{k,s} = \frac{1}{C(n, s)} \left\{ \frac{1}{|a_s|} \right\}_{k+1} R^{s+1} + \frac{1}{C(n, s)} \left\{ \frac{1}{|a_s|} \right\}_{k+s+1} R^{s+1}.
\]
The result is best possible for \( s = 1 \) and equality in (1.7) holds for \( p(z) = (z + k)^n \).

**Remark 1.1.** Taking limit as \( \lambda \to 1 \), our theorem reduces to the following interesting result, which gives a generalization of the result of Mir [13].

**Corollary 1.1.** If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having no zero in \( |z| < k, \ k > 0 \), then for \( 0 < r \leq R \leq k \) and \( 1 \leq s \leq n \),
\[
\max_{|z|=R} |p^{(s)}(z)| \leq \frac{n(n-1)\ldots(n-s+1)}{R^n + \phi_{k,s}} \left[ \left| \frac{R+k}{r+k} \right| \max_{|z|=R} |p(z)| - \min_{|z|=R} |p(z)| \right],
\]
where
\[
\phi_{k,s} = \frac{1}{C(n, s)} \left\{ \frac{1}{|a_s|} \right\}_{k+1} R^{s+1} + \frac{1}{C(n, s)} \left\{ \frac{1}{|a_s|} \right\}_{k+s+1} R^{s+1}.
\]

**Remark 1.2.** Putting \( R = r = 1 \), Corollary 1.1 further reduces to the result proved by Mir [13], namely

**Corollary 1.2.** If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having no zero in \( |z| < k, \ k \geq 1 \), then for \( 1 \leq s \leq n \),
\[
\max_{|z|=1} |p^{(s)}(z)| \leq \frac{n(n-1)\ldots(n-s+1)}{(k+1)^n} \left[ \left| \frac{R+k}{r+k} \right| \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right],
\]
where
\[
\psi_{k,s} = \frac{1}{C(n, s)} \left\{ \frac{1}{|a_s|} \right\}_{k+1} R^{s+1} + \frac{1}{C(n, s)} \left\{ \frac{1}{|a_s|} \right\}_{k+s+1} R^{s+1}.
\]

**Remark 1.3.** Further in Corollary 1.2, it is seen similar in the proof of the theorem that for a complex number \( \lambda \) with \( |\lambda| < 1 \), polynomial \( p(z) - \lambda m \), where \( m = \min_{|z|=k} |p(z)| \), has no zero in \( |z| < k, \ k \geq 1 \) and it follows by applying inequality (2.2) of Lemma 2.1 to \( p(Rz) - \lambda m \), that
By Lemma 2.3, we get
\[ C(n, s)|a_0 - \lambda m| \geq |a_s k| , \]
Letting \( \lambda \to 1 \), we get
\[ C(n, s) \left( |a_0| - |\lambda m| \right) \geq |a_s| k^s . \]
which in turn implies \( \psi_{k,s} \) of Corollary 1.2 is such that \( \psi_{k,s} \geq k^s \) for \( 1 \leq s \leq n \). This implies that Corollary 1.2 is an improvement of a result of Govil [8].

**Remark 1.4.** Putting \( \lambda = 0 \), our theorem reduces to the following result, which is a generalization of a result of Aziz and Rather [2].

**Corollary 1.3.** If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree \( n \) having no zero in \( |z| < k, k > 0 \), then
\[ \max_{|z|=R} |p^{(s)}(z)| \leq R^{s+s} \alpha_{k,s} \max_{|z|=r} |p(z)| \left\{ \begin{array}{l} \left( \frac{R+k}{r+k} \right)^{s-s} \max_{|z|=k} |p(z)| R^{s-s} \min_{|z|=k} |p(z)| \end{array} \right\}. \tag{1.8} \]
where
\[ \alpha_{k,s} = \frac{k^{s+1} + \frac{1}{C(n,s)} |a_0| k^r R}{R + \frac{1}{C(n,s)} |a_0| k^{s+1}} . \]

**Remark 1.5.** If we put \( R = r = 1 \) in Corollary 1.1 then in this case \( k \geq 1 \) and by Lemma 2.1, we have
\[ \left( k-1 \right) \left\{ 1 - \frac{1}{C(n,s)} |a_0| k^s \right\} \geq 0 , \]
which is equivalent to
\[ k^{s+1} + \frac{1}{C(n,s)} |a_0| k^r R \geq k^s . \tag{1.9} \]
Inequality (1.9) clearly shows that Corollary 1.1 is an improvement and a generalization of a result proved by Govil and Rahman [7].

**Remark 1.5.** Corollary 1.1 provides a generalization of a result due Aziz and Rather [2].

**Remark 1.5.** If we assign \( \lambda = 0 \) and \( R = r = s = 1 \), our theorem reduces to a result of Govil et al [8].

**Remark 1.6.** Corollary 1.2 is an improved as well as a generalised version of a well-known inequality proved by Malik [11] under the same set of hypotheses.

**Remark 1.7.** If we put \( s = r = R = k = 1 \), Corollary 1.1 gives an improvement of inequality (1.2), conjectured by Erdös and later proved by Lax [10].
II LEMMA

The following lemmas are needed for the proof of the theorem.

Lemma 2.1. If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree \( n \) such that \( p(z) \neq 0 \) for \( |z| < k, k \geq 1 \), then

\[
\mu_{k,s} |p^{(s)}(z)| \leq |q^{(s)}(z)| \text{ for } |z| = 1, \tag{2.1}
\]

and

\[
\frac{1}{C(n,s)_{k_{0}}} \leq 1, \tag{2.2}
\]

where

\[
q(z) = z^{n} p\left(\frac{1}{z}\right),
\]

\[
k, s = \begin{cases}
\text{for } C(n,s)_{k_{0}} & \text{and } C(n,s) \text{ is as defined in our theorem.}
\end{cases}
\]

The above result is due to Aziz and Rather [2].

Lemma 2.2. If \( p(z) = a_{0} + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu}, 1 \leq \mu \leq n, \) is a polynomial of degree \( n \) such that \( p(z) \neq 0 \) in \( |z| < k, k > 0 \), then for \( 0 < r \leq R \leq k \),

\[
\max_{|z|=R} |p(z)| \leq \left(\frac{R^{n+k}}{r^{n+k}} + k^{2s}\right)^{\mu} \max_{|z|=r} |p(z)| - \left(\left(\frac{R+k}{r^{\mu}} + k^{2s}\right)^{\mu} - 1\right) \min_{|z|=k} |p(z)|, \tag{2.3}
\]

Equality holds in (2.2) for \( p(z) = \left(z^{n+k}\right)^{\mu} \) where \( n \) is a multiple of \( \mu \).

This Lemma is due to Dewan et al [5].

Lemma 2.3. If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree \( n \) with \( p(z) \neq 0 \) for \( |z| < k, k \geq 1 \), then \( |p(z)| > m \) for \( |z| < k \), and in particular \( |a| > m \), where \( m = \min_{|z|=k} |p(z)| \).

The above lemma is due to Gardner et al. [6].

The following lemma was proved by Govil [7].

Lemma 2.4. If \( p(z) \) is a polynomial of degree \( n \) having no zero in \( |z| < k, k \geq 1 \), then for \( |z| \geq \frac{1}{k} \),

\[
|q^{(s)}(z)| \geq mn(n-1)...(n-s+1)z^{n-s}, \tag{2.4}
\]

where \( m = \min_{|z|=k} |p(z)| \) and \( q(z) = z^{n} p\left(\frac{1}{z}\right) \).

Lemma 2.5. The function
\[ \phi(x) = k^{x+1} \left| 1 + \frac{1}{C(n, s)} \left| \frac{a_s}{n} \right| k^{s-1} \right| \left| 1 + \frac{1}{C(n, s)} \left| \frac{a}{x} \right| k^{s+1} \right|, \quad k \geq 1 \]

is an increasing function of \( x \).

\( C(n, s) \) is as defined in our theorem.

**Proof of Lemma 2.5.** The proof follows easily by considering the first derivative test on \( \phi(x) \).

III. PROOF OF THE THEOREM

If \( p(z) \) has no zero in \([k] \), \( k > 0 \) and if \( 0 < r \leq R \leq k \), then \( P(z) = p(Rz) \) has no zero in \( |z| < \frac{k}{R} \), \( \frac{k}{R} \geq 1 \). Now, \( m = \min_{|z|=k} |P(z)| = \min_{|z|=k} |p(Rz)| = \min_{|z|=k} |p(z)| \) .

If \( m > 0 \), then it follows from Rouche’s Theorem that for every real or complex number \( \lambda \) with \( |\lambda| < 1 \) the polynomial \( F(z) = P(z) - \lambda m \) has no zero in \( |z| < \frac{k}{R} \), \( \frac{k}{R} \geq 1 \). Applying inequality (2.1) of Lemma 2.1 to \( F(z) \), we have for \( |z| = 1 \)

\[ k_{s,s} \left| P^{(s)}(z) \right| \leq \left| Q^{(s)}(z) - \lambda mn(n-1)...(n-s+1)z^{x-s} \right|, \quad (3.1) \]

where

\[ Q(z) = z^n P \left( \frac{1}{z} \right), \]

\[ k_{s,s} = \frac{1}{R} \left\{ k^{x+1} + \frac{1}{C(n, s)} \left| a_s \right| k^{s-1} \right\}, \]

and

\[ C(n, s) = \frac{n!}{s!(n-s)!}. \]

Since \( |\lambda| < 1 \), we have

\[ |a_0 - \lambda m| \geq |a_0| - |\lambda| |m|. \quad (3.2) \]

In view of inequality (3.2) and Lemma 2.5, inequality (3.1) gives for \( |z| = 1 \),

\[ k_{s,s} \left| P^{(s)}(z) \right| \leq \left| Q^{(s)}(z) - \lambda mn(n-1)...(n-s+1)z^{x-s} \right|, \quad (3.3) \]

where

\[ k_{s,s} = \frac{1}{R} \left\{ k^{x+1} + \frac{1}{C(n, s)} \left| a_s \right| k^{s-1} \right\}, \]

If we choose the argument of \( \lambda \) on the right hand side of (3.3) such that for \( |z| = 1 \),

\[ Q^{(s)}(z) - \lambda mn(n-1)...(n-s+1)z^{x-s} \right| = Q^{(s)}(z) - \frac{k}{k} |mn(n-1)...(n-s+1), \quad (3.4) \]
which is possible by inequality (2.3) of Lemma 2.4. Using (3.4) to (3.3), we obtain
\[ \mu_{k,s} \left| P^{(s)}(z) \right| \leq \left| Q^{(s)}(z) \right| - \lambda |mn(n-1)...(n-s+1)|, \text{ for } |z| = 1. \]  
(3.5)

Now, if \( f(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq 1 \), then \( g(z) = z^n f \left( \frac{1}{z} \right) \) has no zero in \( |z| < 1 \). Hence, using inequality (2.1) of Lemma 2.4 with \( k = 1 \), we have for \( |z| = 1 \)
\[ |g^{(s)}(z)| \leq |f^{(s)}(z)|. \]
(3.6)

Let \( M = \max_{|z|=1} |P(z)| \), then for every real or complex number \( \gamma \) with \( |\gamma| > 1 \), we have, by Rouche’s theorem the polynomial \( T(z) = P(z) - \gamma M z^n \) has all its zeros in \( |z| < 1 \). Suppose
\[ S(z) = z^n T \left( \frac{1}{z} \right) = z^n P \left( \frac{1}{z} \right) - \gamma M = Q(z) - \gamma M \]
Applying (3.6) to \( T(z) \), we get for \( 1 \leq s \leq n \) and \( |z| = 1 \)
\[ |S^{(s)}(z)| \leq |T^{(s)}(z)|, \]
which implies
\[ \left| Q^{(s)}(z) \right| \leq \left| P^{(s)}(z) - \gamma M n(n-1)...(n-s+1)z^n \right| \text{ for } |z| = 1. \]  
(3.7)

Since \( P(z) \) is of degree \( n \), it is evident that the polynomial \( P^{(s)}(z) \) is of degree \( (n-s) \) and repeated application of Bernstein’s inequality (1.1) to \( P(z) \) yields for \( |z| = 1 \)
\[ |P^{(s)}(z)| \leq M n(n-1)...(n-s+1). \]  
(3.8)

Further, for suitable choice of the argument of \( \gamma \) in (3.7), we have for \( |z| = 1 \)
\[ \left| Q^{(s)}(z) \right| \leq \left| \gamma M n(n-1)...(n-s+1) \right| - |P^{(s)}(z)|, \]
which becomes by taking limit as \( |\gamma| \to 1 \),
\[ \left| Q^{(s)}(z) \right| \leq M n(n-1)...(n-s+1) - |P^{(s)}(z)|, \]
which for \( |z| = 1 \), becomes on using (3.8) that
\[ |P^{(s)}(z)| + |Q^{(s)}(z)| \leq n(n-1)...(n-s+1)M. \]  
(3.9)

Inequality (3.9) in conjunction with inequality (3.5) gives for \( |z| = 1 \)
\[ \left| P^{(s)}(z) \right| \leq \frac{1}{1 + \mu_{k,s}} n(n-1)...(n-s+1) \left\{ \max_{|z|=1} |P(z)| - \lambda |mn| \right\}, \]  
(3.10)

which on replacing \( P(z) \) by \( p(Rz) \), we have for \( |z| = 1 \),
\[ \left| P^{(s)}(z) \right| \leq \frac{1}{R^s + \delta_{k,s}} n(n-1)...(n-s+1) \left\{ \max_{|z|=1} |p(z)| - \lambda \min_{\|z|=1} |p(z)| \right\} \text{ for } |z| = 1, \]  
(3.11)

where \( \delta_{k,s} \) is as defined in the theorem.

When \( \mu = 1 \), Lemma 2.2 becomes for \( 0 < r \leq R \leq k \),
\[ \max_{|z|=1} |p(z)| \leq \left\{ \max_{|z|=1} \left| \left( \frac{R+k}{r+k} \right)^a \right| \right\} \left| \min_{|z|=1} p(z) \right| + \lambda \min_{|z|=1} |p(z)|. \]  
(3.12)

Using inequality (3.12) to inequality (3.11), we obtain
\[
\max_{|z|=R} |p^{(s)}(z)| \leq \frac{n(n-1)\ldots(n-s+1)}{R^s + \delta_{k,s}} \left( \max_{|z|=R} |p(z)| \right) \left( \frac{R+k}{r+k} \right)^n - 1 + \frac{k^s}{\min_{|z|=k} |p(z)|},
\]

which completes the proof of the theorem.

REFERENCES


