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Rationally Expressed Inequalities And Invariant Point Theorems In Multiplicative Semi-Metric Spaces

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Abstract.
The primary aspect of this manuscript is to describe a invariant point conclusion of functions that satisfy the theory of rationally expressive inequalities in multiplicative semi metric spaces for multiple functions. Common invariant point theorems based on weakly compatible functions with E.A and common limiting value range properties are also discussed. The major objective of the study is to establish the invariant point which is common and unique as well theorems and using the principle of weak compatibility in the Multiplicative semi metric space. Our analysis generalizes the existing results of Multiplicative semi metric space.

Keywords. Multiplicative semi metric spaces, rational inequalities, weakly compatible Functions

AMS Subject Classification. 37C25, 55M20

INTRODUCTION AND PRELIMINARIES

In 1906, the conception of metric space was proposed by Frechet, which is a significant area in the study of science , especially in invariant point theory. Ever As of, metric space has been extended in several contexts, such as quasi-metric space, semi-metric space, etc. Throughout the 17th century, Newton and Leibnitz established a most significant principles in differential and integral calculus. In 1972, Grossman and Katz [1] presented a multiplicative calculus also known as a non-Newtonian calculus whereas in the year 2008, Bashirov et al.[2] introduced a new metric space called multiplicative semi metric Space. In Non-Newtonian calculus, Ozavsar et al.[3] have studied the notion of convergence and have provided an impression of convergence. Multiplicative Semi metric space is weaker space than Multiplicative semi metric Space. The conceptualization was initiated by Austrian-American mathematician Menger [4]who named this space as Symmetric space. Cicchese[5] thereafter confirmed the effects of the widely known invariant point results by adding contract functions. By generalizing the idea of commuting functions, Jungck [6] proposed the compatible functions in 1988. The compatible functions of type-E were established by Singh et al. [7]. The conception of reciprocally continuous was proposed by Pant[8] in 1999. Numerous invariant point theorems are generated [9,10] using the aforesaid ideas, and the theory of compatible type-E is also extensively used in Menger space [11].In the context of multiplicative metric spaces, Özavsar and Cevikel [3] have shown an equivalent conclusion to the Banach contraction principle.In the approach to differential and integral problems, invariant points that satisfy those contractive conditions have cognitive benefits (see, for example, [23-27]). Invariant point theorems of some mappings formulated several other essential topics such as coincidence points, crossing points, sectional points, etc.

Note that $\mathbb{R}_+$is a complete multiplicative semi-metric space with respect to multiplicative semi-metrics to illustrate the significance of this research. In addition, using the inequality in multiplicative semi-metric space using the instance of multiplicative contraction, we illustrate a common invariant point theorem. There are many applications for the analysis of invariant points of functions that satisfy assertive contraction Specifications and have been at the core of various research activities for many years, even to date. So to establish various invariant point theorems in the framework of multiplicative semi metric space is the core concept of this research article.

Definition 1.1. [2] Let ω be a nonempty set. A multiplicative metric is a mapping $□: \omega \times \omega \rightarrow \mathbb{R}_+$satisfying the following Specifications:

(1.1) $□(\alpha,\beta) \geq 1 \; \forall \; \alpha,\beta \in \omega$ and $□(\alpha,\beta) = 1$ ifand only if $\alpha = \beta$;

(1.2) $□(\alpha,\beta) = □(\beta,\alpha) \; \forall \; \alpha,\beta \in \omega$;

(1.3) $□(\alpha,\beta) \leq □(\alpha,□) + □(□,\beta) \; \alpha,\beta \in \omega$ (multiplicative triangular inequality).

The pair $(\omega,□)$is called Multiplicative semi metric Space.

Definition 1.2. [7] Let ω be a nonempty set. A multiplicative metric is a mapping $□: \omega \times \omega \rightarrow \mathbb{R}_+$satisfying the following specifications:

(1.4) $□(\alpha,\beta) \geq 1 \; \forall \; \alpha,\beta \in \omega$ and $□(\alpha,\beta) = 1$ ifand only if $\alpha = \beta$;

(1.5) $□(\alpha,\beta) = □(\beta,\alpha) \forall \; \alpha,\beta \in \omega$.

The pair (ω, □) is called Multiplicative Semi metric space.

Definition 1.3. [7] Two self-maps ζ and □ of MSMS $(\omega,□)$ are said to be weakly compatible if they commute at coincidence points, i.e. if $\zeta\tau = □\tau$for an arbitrary$\tau \in \omega$ implies $\zeta□\tau = □\zeta\tau$.

Definition 1.4. Letζ and□be two functions of a multiplicative semi metric space$(\omega,□)$into itself, then ζ and□are known to be

(1.6) *commutative mapping* if $\zeta□\alpha = □\zeta\alpha \; \forall \; \alpha \in \omega$.

(1.7) *Weak commutative mapping* if $d(\zeta□\alpha,□\zeta\alpha) \leq d(\zeta\alpha,□\alpha) \; \forall \; \alpha \in \omega$.

(1.8) *E.A property* if there is an existence of a sequence $\{\alpha_\varsigma\}$ in $\omega$and that is how that$\lim_{\varsigma\to\infty} \zeta\alpha_\varsigma$=$\lim_{\varsigma\to\infty} □\alpha_\varsigma = \tau$ for an arbitrary $\tau \in \omega$.

(1.9) *CLR□property* (common limiting value range of □ property) if there is an existence of a sequence $\{\alpha_\varsigma\}$ in $\omega$and that is how that$\lim_{\varsigma\to\infty} \zeta\alpha_\varsigma$=$\lim_{\varsigma\to\infty} □\alpha_\varsigma = □\tau$ for an arbitrary $\tau \in \omega$.

(1.10) CLR_ζ*property* (common limiting value range of ζproperty) if there is an existence of a sequence $\{\alpha_\varsigma\}$ in $\omega$and that is how that$\lim_{\varsigma\to\infty} \zeta x_\varsigma$=$\lim_{\varsigma\to\infty} □\alpha_\varsigma = \zeta\tau$ for an arbitrary $\tau \in \omega$.

2. Main results

In 2014, He et al. [4] established the common invariant point theorem for pairs ofweakly commuting functions in a multiplicative semi metric space as follows:

Theorem 2.1 Supposeζ, □, □ and □ are functions of a complete multiplicative semi metric Space$(\omega,□)$ into itself satisfying the following Specifications:

$(\Psi 1)$ $□\omega \subset □\omega, □\omega \subset \zeta\omega$;

$(C2)□(□\alpha,□\beta) \leq [max\{□(\zeta\alpha,□\beta),□(\zeta\alpha,□\alpha),□(□\beta,□\beta),□(□\alpha,□\beta),□(\zeta\alpha,□\beta)\}]^\varphi$

for all $\alpha,\beta \in \omega$,where $\varphi \in (0,1/2)$.

Suppose that any one of the functionsζ, □, □ and □ is continuous, and the pairs ζ, □ and □, □ are weakly commuting. Then ζ, □, □ and □ have a invariant Point which is common and unique as well.

We now specify the following theorems for weakly compatible functions in a multiplicative semi-metric space employing rationally expressive inequality as continues to follow:

Theorem 2.2 Supposeζ, □, □ and □ are functions of a complete multiplicative semi metric space$(\omega, □)$ into itself satisfying the specifications

$(\Psi 1)$ $□\omega \subset □\omega, □\omega \subset \zeta\omega$

$(\Psi 2)$

$$□(□\alpha, □\beta) \leq \left\{ max \left\{ \frac{□(\zeta\alpha, □\beta)[□(\zeta\alpha, □\alpha) + □(□\beta, □\alpha)]}{□(□\beta, □\beta) + □(□\beta, \zeta\alpha)}, \frac{□(\zeta\alpha, □\alpha)[□(□\beta, □\beta) + □(□\beta, \zeta\alpha)]}{□(\zeta\alpha, □\alpha) + □(□\beta, □\alpha)} \right\} \right\}^{\varphi}$$

for all $\alpha, \beta \in \omega$, where $\varphi \in (0,1/2)$

(Ψ3) Presumed that the pairs $(\zeta, □)$ and $(□, □)$ are weakly compatible.

(Ψ4) One of the subspaces $\zeta\omega$ or $□\omega$ or $□\omega$ or $□\omega$ is complete.

Then ζ, □, □ and □ have a invariant Point which is common and unique as well.

*Proof.*Suppose$\alpha_0 \in \omega$be a point which is arbitrarily chosen. As of$□\omega \subset □\omega$,there is the possibility of$\alpha_1 \in \omega$and that is how that$□\alpha_0 = □\alpha_1 = \beta_0$. Now for this $\alpha_1$there is the possibility of$\alpha_2 \in \omega$and that is how that$□\alpha_1 = \zeta\alpha_2 = \beta_1$. In the same manner, we can inductively define a sequence $\{\alpha_\varsigma\}$and that is how that$□\alpha_{2\varsigma} = □\alpha_{2\varsigma+1} = \beta_{2\varsigma}$; $□\alpha_{2\varsigma+1} = \zeta\alpha_{2\varsigma+2} = \beta_{2\varsigma+1}$,

Using $(\Psi 2)$, we've got

$□(\beta_{2\varsigma}, \beta_{2\varsigma+1}) \leq □(□\alpha_{2\varsigma}, □\alpha_{2\varsigma+1})$

$$\leq \left\{ max \left\{ \frac{□(\zeta\alpha_{2\varsigma}, □\alpha_{2\varsigma+1})[□(\zeta\alpha_{2\varsigma}, □\alpha_{2\varsigma}) + □(□\alpha_{2\varsigma+1}, □\alpha_{2\varsigma})]}{□(□\alpha_{2\varsigma+1}, □\alpha_{2\varsigma+1}) + □(□\alpha_{2\varsigma+1}, \zeta\alpha_{2\varsigma})}, \frac{□(\zeta\alpha_{2\varsigma}, □\alpha_{2\varsigma})[□(□\alpha_{2\varsigma+1}, □\alpha_{2\varsigma+1}) + □(□\alpha_{2\varsigma+1}, \zeta\alpha_{2\varsigma})]}{□(\zeta\alpha_{2\varsigma}, □\alpha_{2\varsigma}) + □(□\alpha_{2\varsigma+1}, □\alpha_{2\varsigma})} \right\} \right\}^{\varphi}$$

$$\leq \left\{ max \left\{ \frac{□(\beta_{2\varsigma-1}, \beta_{2\varsigma})[□(\beta_{2\varsigma-1}, \beta_{2\varsigma}) + □(\beta_{2\varsigma+1}, \beta_{2\varsigma})]}{□(\beta_{2\varsigma}, \beta_{2\varsigma+1}) + □(\beta_{2\varsigma}, \beta_{2\varsigma-1})}, \frac{□(\beta_{2\varsigma-1}, \beta_{2\varsigma})[□(\beta_{2\varsigma}, \beta_{2\varsigma+1}) + □(\beta_{2\varsigma}, \beta_{2\varsigma-1})]}{□(\beta_{2\varsigma-1}, \beta_{2\varsigma}) + □(\beta_{2\varsigma+1}, \beta_{2\varsigma})} \right\} \right\}^{\varphi}$$

$$\leq \left\{ max \left\{ \begin{array}{c} \frac{\square(\beta_{2\varsigma-1}, \beta_{2\varsigma})[\square(\beta_{2\varsigma-1}, \beta_{2\varsigma}) + \square(\beta_{2\varsigma+1}, \beta_{2\varsigma})]}{\square(\beta_{2\varsigma}, \beta_{2\varsigma+1}) + \square(\beta_{2\varsigma}, \beta_{2\varsigma-1})}, \\ \frac{\square(\beta_{2\varsigma-1}, \beta_{2\varsigma})[\square(\beta_{2\varsigma}, \beta_{2\varsigma+1}) + \square(\beta_{2\varsigma}, \beta_{2\varsigma-1})]}{\square(\beta_{2\varsigma-1}, \beta_{2\varsigma}) + \square(\beta_{2\varsigma+1}, \beta_{2\varsigma})} \end{array} \right\} \right\}^{\varphi}$$

$$\leq \left\{ max \left\{ \begin{array}{c} \square(\beta_{2\varsigma-1}, \beta_{2\varsigma}), \\ \square(\beta_{2\varsigma-1}, \beta_{2\varsigma}) \end{array} \right\} \right\}^{\varphi}$$

$$\square(\beta_{2\varsigma}, \beta_{2\varsigma+1}) \leq \square^{\frac{\varphi}{1-\varphi}}(\beta_{2\varsigma-1}, \beta_{2\varsigma})$$

$\square(\beta_{2\varsigma}, \beta_{2\varsigma+1}) \leq \square^{\square}(\beta_{2\varsigma-1}, \beta_{2\varsigma})$Where $\square = \frac{\varphi}{1-\varphi}$

In a similar way we've got,

$\square(\beta_{2\varsigma+1}, \beta_{2\varsigma+2}) = \square(\square\alpha_{2\varsigma+1}, \square\alpha_{2\varsigma+2}) = \square(\square\alpha_{2\varsigma+2}, \square\alpha_{2\varsigma+1})$

$$\leq \left\{ max \left\{ \begin{array}{c} \frac{\square(\zeta\alpha_{2\varsigma+2}, \square\alpha_{2\varsigma+1})[\square(\zeta\alpha_{2\varsigma+2}, \square\alpha_{2\varsigma+2}) + \square(\square\alpha_{2\varsigma+1}, \square\alpha_{2\varsigma+2})]}{\square(\square\alpha_{2\varsigma+1}, \square\alpha_{2\varsigma+1}) + \square(\square\alpha_{2\varsigma+1}, \zeta\alpha_{2\varsigma+2})}, \\ \frac{\square(\zeta\alpha_{2\varsigma+2}, \square\alpha_{2\varsigma+2})[\square(\square\alpha_{2\varsigma+1}, \square\alpha_{2\varsigma+1}) + \square(\square\alpha_{2\varsigma+1}, \zeta\alpha_{2\varsigma+2})]}{\square(\zeta\alpha_{2\varsigma+2}, \square\alpha_{2\varsigma+2}) + \square(\square\alpha_{2\varsigma+1}, \square\alpha_{2\varsigma+2})} \end{array} \right\} \right\}^{\varphi}$$

$$\leq \left\{ max \left\{ \begin{array}{c} \frac{\square(\beta_{2\varsigma+1}, \beta_{2\varsigma})[\square(\beta_{2\varsigma+1}, \beta_{2\varsigma+2}) + \square(\beta_{2\varsigma+1}, \beta_{2\varsigma+2})]}{\square(\beta_{2\varsigma}, \beta_{2\varsigma+1}) + \square(\beta_{2\varsigma}, \beta_{2\varsigma+1})}, \\ \frac{\square(\beta_{2\varsigma+1}, \beta_{2\varsigma+2})[\square(\beta_{2\varsigma}, \beta_{2\varsigma+1}) + \square(\beta_{2\varsigma}, \beta_{2\varsigma+1})]}{\square(\beta_{2\varsigma+1}, \beta_{2\varsigma+2}) + \square(\beta_{2\varsigma+1}, \beta_{2\varsigma+2})} \end{array} \right\} \right\}^{\varphi}$$

$$\leq \left\{ max \left\{ \begin{array}{c} \square(\beta_{2\varsigma+1}, \beta_{2\varsigma+2}), \\ \square(\beta_{2\varsigma}, \beta_{2\varsigma+1}) \end{array} \right\} \right\}^{\varphi}$$

$\square(\beta_{2\varsigma+1}, \beta_{2\varsigma+2}) \leq \square^{\varphi}(\beta_{2\varsigma}, \beta_{2\varsigma+1}). \square^{\varphi}(\beta_{2\varsigma+1}, \beta_{2\varsigma+2})$ [Using Symmetry and (ii)]

Which involves ensuring that,

$\square(\beta_{2\varsigma+1}, \beta_{2\varsigma+2}) \leq \square^{\frac{\varphi}{1-\varphi}}(\beta_{2\varsigma}, \beta_{2\varsigma+1})$

On substituting, h = $\frac{\varphi}{1-\varphi} \in (0, \frac{1}{2})$

$\square(\beta_{2\varsigma+1}, \beta_{2\varsigma+2}) \leq \square^{\square}(\beta_{2\varsigma}, \beta_{2\varsigma+1})$,

Consequently

$$\square(\beta_{\varsigma}, \beta_{\varsigma+1}) \leq \square^{\square^1}(\beta_{\varsigma-1}, \beta_{\varsigma}) \leq \square^{\square^2}(\beta_{\varsigma-2}, \beta_{\varsigma-1}) \leq \cdots \leq \square^{\square^{\varsigma}}(\beta_0, \beta_1)$$

For all $\varsigma \geq 2$,Suppose$\sigma, \varsigma \in \mathbb{N}$and that is how that$\sigma \geq \varsigma$. Using the triangular multiplicative inequality, we obtain

$$\begin{aligned} \square(\beta_{\sigma}, \beta_{\varsigma}) &\leq \square(\beta_{\sigma}, \beta_{\sigma-1}). \square(\beta_{\sigma-1}, \beta_{\sigma-2}) \ldots \square(\beta_{\varsigma+1}, \beta_{\varsigma}) \\ &\leq \square^{\square^{\sigma-1}}(\beta_1, \beta_0). \square^{\square^{\sigma-2}}(\beta_1, \beta_0) \ldots \square^{\square^{\varsigma}}(\beta_1, \beta_0) \\ &\leq \square^{\frac{\square^{\varsigma}}{1-\square}}(\beta_1, \beta_0) \end{aligned}$$

Supposelimiting value as $\sigma, \varsigma \to \infty$,we've got$\square(\beta_{\sigma}, \beta_{\varsigma}) \to 1$. Therefore $\{\beta_{\varsigma}\}$ is a multiplicative Cauchy sequence.Now, suppose that $\zeta\omega$ is complete there is the possibility of$\square \in \zeta\omega$and that is how that $\beta_{2\varsigma+1} = \square\alpha_{\varsigma+1} = \zeta\alpha_{2\varsigma+2} \to \square$ as $\varsigma \to \infty$

Consequently, we can find $□ \in \omega$and that is how that$\zeta □ = □$.Further a multiplicative Cauchy sequence $\{\beta_{2\varsigma+1}\}$ has a convergent subsequence $\{ \beta_{2\varsigma+1}\}$,As a consequence,

The sequence $\{\beta_{\varsigma}\}$ converges and Consequently a subsequence $\{\beta_{2\varsigma}\}$ also converges.

Thus we've got,

$$\beta_{2\varsigma} = □\alpha_{2\varsigma} = □\alpha_{2\varsigma+1} \to □ \; as \; \varsigma \to \infty$$

Now we're claiming$□□ = □$.On substituting$\alpha = □$ and $\beta = \alpha_{2\varsigma+1}$in inequality$(C2)$, we get

$$□(□□, \beta_{2\varsigma+1}) = □(□□, □\alpha_{2\varsigma+1})$$

$$\leq \left\{ max \left\{ \frac{□\left(\zeta □, □\alpha_{2\varsigma+1}\right)\left[□(\zeta □, □□) + □\left(□\alpha_{2\varsigma+1}, □□\right)\right]}{□\left(□\alpha_{2\varsigma+1}, □\alpha_{2\varsigma+1}\right) + □\left(□\alpha_{2\varsigma+1}, \zeta □\right)}, \frac{□(\zeta □, □□)\left[□\left(□\alpha_{2\varsigma+1}, □\alpha_{2\varsigma+1}\right) + □(□\alpha_{2\varsigma+1}, \zeta □)\right]}{□(\zeta □, □□) + □(□\alpha_{2\varsigma+1}, □□)} \right\} \right\}^{\varphi}$$

Consideringlimiting value$\varsigma \to \infty$, we've got

$$□(□□, □) \leq \left\{ max \left\{ \frac{□(□, □)[□(□, □□) + □(□, □□)]}{□(□, □) + □(□, □)}, \frac{□(□, □□)[□(□, □) + □(□, □)]}{□(□, □□) + □(□, □□)} \right\} \right\}^{\varphi} \leq \{max\{□(□, □□), 1\}\}^{\varphi}$$

$$= \{□(□, □□)\}^{\varphi}$$

Which ensures that$□(□□, □) = 1$ and consequently$□ = □□ = \zeta □$. As a consequence, □ is aCoincidence point of ζ and □.As of$□ = □□ \in □\omega \subset □\omega$there is the possibility of$\mu \in \omega$and that is how that$□ = □\mu$.

Next it is to be claimed that$□\mu = □$. Now

$$□(□, □\mu) = □(□\alpha_{2\varsigma}, □\mu)$$

$$\leq \left\{ max \left\{ \frac{□\left(\zeta\alpha_{2\varsigma}, □\mu\right)\left[□\left(\zeta\alpha_{2\varsigma}, □\alpha_{2\varsigma}\right) + □\left(□\mu, □\alpha_{2\varsigma}\right)\right]}{□(□\mu, □\mu) + □\left(□\mu, \zeta\alpha_{2\varsigma}\right)}, \frac{□\left(\zeta\alpha_{2\varsigma}, □\alpha_{2\varsigma}\right)[□(□\mu, □\mu) + □(□\mu, \zeta\alpha_{2\varsigma})]}{□\left(\zeta\alpha_{2\varsigma}, □\alpha_{2\varsigma}\right) + □(□\mu, □\alpha_{2\varsigma})} \right\} \right\}^{\varphi}$$

Consideringlimiting value$\varsigma \to \infty$, we've got

$$□(□, □□) \leq \left\{ max \left\{ \frac{□(□, □)[□(□, □) + □(□\mu, □)]}{□(□, □\mu) + □(□, □)}, \frac{□(□, □)[□(□, □□) + □(□, □)]}{□(□, □) + □(□□, □)} \right\} \right\}^{\varphi}$$

$$\leq \{max\{1,1\}\}^{\varphi}$$

$$= \{□(□, □\mu)\}^{\varphi}$$

Which ensures that$□(□, □\mu) = 1$ and Consequently$□ = □\mu = □\mu$.As a consequence,μ is aCoincidence point of □and □. As of the pairs $\zeta, □$ and $□, □$ are weakly compatible, we've got

$$□□ = □(\zeta □) = \zeta(□□) = \zeta □; \; □□ = □(□\mu) = □(□\mu) = □□.$$

Next it is to be claimed that$\square\square = \square$. On substituting$\alpha = \square$ and $\beta = \alpha_{2\varsigma+1}$in $(\Psi 2)$, we've got

$$\square(\square\square, \square\alpha_{2\varsigma+1}) \leq \left\{ max \left\{ \begin{array}{c} \dfrac{\square\left(\zeta\square, \square\alpha_{2\varsigma+1}\right)\left[\square(\zeta\square, \square\square) + \square\left(\square\alpha_{2\varsigma+1}, \square\square\right)\right]}{\square\left(\square\alpha_{2\varsigma+1}, \square\alpha_{2\varsigma+1}\right) + \square\left(\square\alpha_{2\varsigma+1}, \zeta\square\right)}, \\ \dfrac{\square(\zeta\square, \square\square)\left[\square\left(\square\alpha_{2\varsigma+1}, \square\alpha_{2\varsigma+1}\right) + \square(\square\alpha_{2\varsigma+1}, \zeta\square)\right]}{\square(\zeta\square, \square\square) + \square(\square\alpha_{2\varsigma+1}, \square\square)} \end{array} \right\} \right\}^{\varphi}$$

Consideringlimiting value$\varsigma \to \infty$, we've got

$$\square(\square\square, \square) \leq \left\{ max \left\{ \begin{array}{c} \dfrac{\square(\square\square, \square)[\square(\square\square, \square\square) + \square(\square, \square\square)]}{\square(\square, \square) + \square(\square, \square\square)}, \\ \dfrac{\square(\square, \square\square)[\square(\square, \square) + \square(\square, \square\square)]}{\square(\square\square, \square\square) + \square(\square, \square\square)} \end{array} \right\} \right\}^{\varphi}$$

$$\square(\square\square, \square) \leq \left\{max\{\square((\square\square, \square), \square((\square, \square\square)\}\right\}^{\varphi} \leq \{\square(\square\square, \square)\}^{\varphi}$$

Which ensures that$\square\square = \square$ and Consequently$\square\square = \zeta\square = \square$. Next it is to be claimed that$\square\square = \square$.On substituting$\alpha = \alpha_{2\varsigma}$and $\beta = \square$ in $(\Psi 2)$;we've got

$$\square(\square\alpha_{2\varsigma}, \square\square) \leq \left\{ max \left\{ \begin{array}{c} \dfrac{\square\left(\zeta\alpha_{2\varsigma}, \square\square\right)\left[\square\left(\zeta\alpha_{2\varsigma}, \square\alpha_{2\varsigma}\right) + \square\left(\square\square, \square\alpha_{2\varsigma}\right)\right]}{\square(\square\square, \square\square) + \square\left(\square\square, \zeta\alpha_{2\varsigma}\right)}, \\ \dfrac{\square\left(\zeta\alpha_{2\varsigma}, \square\alpha_{2\varsigma}\right)[\square(\square\square, \square\square) + \square(\square\square, \zeta\alpha_{2\varsigma})]}{\square\left(\zeta\alpha_{2\varsigma}, \square\alpha_{2\varsigma}\right) + \square(\square\square, \square\alpha_{2\varsigma})} \end{array} \right\} \right\}^{\varphi}$$

Consideringlimiting value$\varsigma \to \infty$,we've got

$$\leq \left\{ max \left\{ \begin{array}{c} \dfrac{\square(\square, \square\square)[\square(\square, \square) + \square(\square\square, \square)]}{\square(\square\square, \square\square) + \square(\square\square, \square)}, \\ \dfrac{\square(\square, \square)[\square(\square\square, \square\square) + \square(\square\square, \square)]}{\square(\square, \square) + \square(\square\square, \square)} \end{array} \right\} \right\}^{\varphi} \leq \left\{max\{1,1\}\right\}^{\varphi}$$

Which ensures that$\square\square = \square$ and Consequently$\square\square = \square\square = \square$.As a consequence,$\square$ is a commoninvariant point of $\zeta, \square, \square$ and $\square$.

In the same manner, we can complete the proofs for cases in which $\square\omega$ or $\square\omega$ or $\square\omega$ is complete.

The distinctiveness can conveniently be deduced from$(\Psi 2)$.This completes the proof.

In Theorem 2.2, if we put $\square = \square$, then we obtain the following corollary.

Corollary2.3 Suppose$\zeta, \square$ and $\square$are functions of a multiplicative semi metric space $(\omega, \square)$ into itself satisfying the Specifications

$(\Psi 1)$ $\square\omega \subset \square\omega, \square\omega \subset \zeta\omega$

$$(\Psi 2)\ \square(\square\alpha, \square\beta) \leq \left\{ max \left\{ \begin{array}{c} \dfrac{\square(\zeta\alpha, \square\beta)[\square(\zeta\alpha, \square\alpha) + \square(\square\beta, \square\alpha)]}{\square(\square\beta, \square\beta) + \square(\square\beta, \zeta\alpha)}, \\ \dfrac{\square(\zeta\alpha, \square\alpha)[\square(\square\beta, \square\beta) + \square(\square\beta, \zeta\alpha)]}{\square(\zeta\alpha, \square\alpha) + \square(\square\beta, \square\alpha)} \end{array} \right\} \right\}^{\varphi}$$

for all $\alpha, \beta \in \omega$, where $\varphi \in (0,1/2)$

$(\Psi 3)$Presumed that the pairs ζ, □ and □, □ are weakly compatible;

(Ψ4) One of the subspaces $\zeta\omega$ or □ω or □ω is complete.

In Theorem 2.2, if we put $\zeta = □ = I$, then we obtain the following corollary.

Corollary 2.4 Suppose□ and □are functions of a complete multiplicative semi metricspace $(\omega, □)$ into itself satisfying

$$(\Psi 5)□(□\alpha, □\beta) \leq \left\{ max \left\{ \begin{array}{c} \frac{□(\alpha, □\beta)[□(\alpha, □\alpha) + □(□\beta, □\alpha)]}{□(\beta, □\beta) + □(\beta, \alpha)}, \\ \frac{□(\alpha, □\alpha)[□(\beta, □\beta) + □(\beta, \alpha)]}{□(\alpha, □\alpha) + □(□\beta, □\alpha)} \end{array} \right\} \right\}^{\varphi}$$

for all $\alpha, \beta \in \omega$,where $\varphi \in (0,1/2)$

$(\Psi 6)$ One of the subspace □ω or □ω is complete.

Then □ and □ have a invariant Point which is common and unique as well.Also we establish the following theorem for weakly compatible functions in a multiplicative semi metric space under the suitable condition dropping the condition of completeness of subspaces as follows :

Theorem 2.5 Supposeζ, □, □ and □are functions of a complete multiplicative semi metric space $(\omega, □)$ satisfying the Specifications$(\Psi 1), (\Psi 2), (\Psi 3)$ and

$(\Psi 5)$ One of the subspaces $\zeta\omega$ or□ωor □ω or □ω is subset which is closed of ω.

Then ζ, □, □ and □ have a invariant Point which is common and unique as well.

Proof. In fact, the subspace of a complete multiplicative semi metric space is complete if and only if it is closed. By Theorem 2.2, this conclusion holds.

This is concluding the statement.

Theorem 2.7 Supposeζ, □, □ and □are functions of a multiplicative metricspace $(\omega, □)$ satisfying the Specifications$(\Psi 1), (\Psi 2), (\Psi 3), (\Psi 5)$ and the followingcondition,

$(\Psi 6)$the pairs ζ, □ and □, □Satisfies the E.A. property.

Then ζ, □, □ and □ have a invariant Point which is common and unique as well.

Proof. Suppose that the pair ζ, □Satisfies the E.A property. Then there is the possibility of a sequence $\{\alpha_\varsigma\}$in $\omega$and that is how that$\lim_{\varsigma\to\infty} A\alpha_\varsigma = \lim_{\varsigma\to\infty} □\alpha_\varsigma = □$for an arbitrary$□ \in \omega$. As of□$\omega \subset □\omega$, there is the possibility of a sequence $\{\beta_\varsigma\}$ in $\omega$and that is how that□$\alpha_\varsigma = □\beta_\varsigma$.Consequently$\lim_{\varsigma\to\infty} □\beta_\varsigma = □$. Also, □$\omega \subset \zeta\omega$ so there is an existence of a sequence $\{□_\varsigma\}$ in $\omega$and that is how that□$□_\varsigma = \zeta\alpha_\varsigma$.Consequently$\lim_{\varsigma\to\infty} □□_\varsigma = □$.Now probably □$\omega$ is a subset which is closed of ω. Then there is the possibility of$\rho \in \omega$and that is how that□ $= □\rho$.Subsequentlywe've got$\lim_{\varsigma\to\infty} □\alpha_\varsigma = \lim_{\varsigma\to\infty} \zeta\alpha_\varsigma = \lim_{\varsigma\to\infty} □□_\varsigma = \lim_{\varsigma\to\infty} □\beta_\varsigma = □\rho$for an arbitrary$\rho \in \omega$. First, it is to be claimed that□$\rho = □$, on substituting $\alpha = \alpha_\varsigma$ and $\beta = \rho$, we've got

$$\square(\square\alpha\varsigma,\square\rho)\leq\left\{max\left\{\begin{array}{c}\frac{\square(\zeta\alpha_{\varsigma},\square\rho)[\square(\zeta\alpha_{\varsigma},\square\alpha_{\varsigma})+\square(\square\rho,\square\alpha_{\varsigma})]}{\square(\square\rho,\square\rho)+\square(\square\rho,\zeta\alpha_{\varsigma})},\\ \frac{\square(\zeta\alpha_{\varsigma},\square\alpha_{\varsigma})[\square(\square\rho,\square\rho)+\square(\square\rho,\zeta\alpha_{\varsigma})]}{\square(\zeta\alpha_{\varsigma},\square\alpha_{\varsigma})+\square(\square\rho,\square\alpha_{\varsigma})}\end{array}\right\}\right\}^{\varphi}$$

considering $\varsigma\to\infty$, we've got

$$\square(\square,\square\rho)\leq\left\{max\left\{\begin{array}{c}\frac{\square(\square,\square)[\square(\square,\square)+\square(\square\rho,\square)]}{\square(\square,\square\rho)+\square(\square,\square)},\\ \frac{\square(\square,\square)[\square(\square,\square\rho)+\square(\square,\square)]}{\square(\square,\square)+\square(\square\rho,\square)}\end{array}\right\}\right\}^{\varphi}$$

Which ensures that $\square(\square,\square\rho)=1$ and Consequently $\square=\square\rho=\square\rho$. As of $\square\omega\subset\zeta\omega$. As a consequence, there is the possibility of $\mu\in\omega$ and that is how that $\square\rho=\square=\zeta\mu$.

Next it is to be claimed that $\square\mu=\square$. On substituting $\alpha=\mu$ and $\beta=\rho$ in $(\Psi 2)$, we've got

$$\square(\square\mu,\square)=\square(\square\mu,\square\rho)$$

$$\leq\left\{max\left\{\begin{array}{c}\frac{\square(\zeta\mu,\square\rho)[\square(\zeta\mu,\square\mu)+\square(\square\rho,\square\mu)]}{\square(\square\rho,\square\rho)+\square(\square\rho,\zeta\mu)},\\ \frac{\square(\zeta\mu,\square\mu)[\square(\square\rho,\square\rho)+\square(\square\rho,\zeta\mu)]}{\square(\zeta\mu,\square\mu)+\square(\square\rho,\square\mu)}\end{array}\right\}\right\}^{\varphi}$$

$$\leq\left\{max\left\{\begin{array}{c}\frac{\square(\square,\square)[\square(\square,\square\mu)+\square(\square,\square\mu)]}{\square(\square,\square)+\square(\square,\square)},\\ \frac{\square(\square,\square\mu)[\square(\square,\square)+\square(\square,\square)]}{\square(\square,\square\mu)+\square(\square,\square\mu)}\end{array}\right\}\right\}^{\varphi}$$

$$=\{max\{(\square,\square\mu),1\}\}^{\varphi}$$

$$\square(\square\mu,\square)=\{\square(\square,\square\mu)\}^{\varphi}$$

Which helps to ensure that $\square\mu=\square$ and consequently $\square=\square\mu=\zeta\mu$ so $\zeta\mu=\square\mu=\square\rho=\square\rho=\square$. As of pairs $\zeta,\square$ and $\square,\square$ are weakly compatible, and μ and $\square$ are theirCoincidence point respectively, so we've got

$$\zeta\square=\zeta(\square\mu)=\square(\zeta\mu)=\square\square;\ \square\square=\square(\square\rho)=\square(\square\rho)=\square\square,$$

Now we establish that $\square$ is common invariant point of $\zeta,\square,\square$ and $\square$ for which weestablish that $\square\mu=\square\square$. Now on substituting $\alpha=\mu$ and $\beta=\square$ in $(\Psi 2)$, we've got

$$\square(\square\mu,\square\square)\leq\left\{max\left\{\begin{array}{c}\frac{\square(\zeta\mu,\square\square)[\square(\zeta\mu,\square\mu)+\square(\square\square,\square\mu)]}{\square(\square\square,\square\square)+\square(\square\square,\zeta\mu)},\\ \frac{\square(\zeta\mu,\square\mu)[\square(\square\square,\square\square)+\square(\square\square,\zeta\mu)]}{\square(\zeta\mu,\square\mu)+\square(\square\square,\square\mu)}\end{array}\right\}\right\}^{\varphi}$$

$$\leq\left\{max\left\{\begin{array}{c}\frac{\square(\square\mu,\square\square)[\square(\square\mu,\square\mu)+\square(\square\square,\square\mu)]}{\square(\square\square,\square\square)+\square(\square\square,\square\mu)},\\ \frac{\square(\square\mu,\square\mu)[\square(\square\square,\square\square)+\square(\square\square,\square\mu)]}{\square(\square\mu,\square\mu)+\square(\square\square,\square\mu)}\end{array}\right\}\right\}^{\varphi}\leq\{max\{\square(\square\mu,\square\square),1\}\}^{\varphi}$$

Which ensures that $\square\mu = \square\square$ and consequently $\square = \square\mu = \square\square$ and $\square = \square\square = \square\square$ so □ is a Commoninvariant point of □ and □.

The distinctiveness can conveniently be deduced from $(\Psi 2)$. This is concluding the statement.

Eventually, we describe in a multiplicative semi metric space, the accompanying theorems for characteristics, which are weakly compatible with the common limiting value range property :

Theorem 2.8 Suppose $\zeta, \square, \square$ and □ are functions of a multiplicative semi metric space $(\omega, \square)$ into itself satisfying the Specifications $(\Psi 1)$; $(\Psi 2), (\Psi 3), (\Psi 5)$ and the following condition:

$(\Psi 7)$ The pair $\zeta, \square$ Satisfies CLR_{ζ} property or the pair □; □ Satisfies $CLR_{\square}$ property.

Then $\zeta, \square, \square$ and □ have a invariant point which is common and unique as well.

Proof. If the pair □, □ Satisfies $CLR_{\square}$ property so there is an existence of sequence $\{\alpha_{\varsigma}\}$ in ω $\lim_{\varsigma\to\infty} \square\alpha_{\varsigma} = \lim_{\varsigma\to\infty} \square\alpha_{\varsigma} = \square \in \square\omega$. As of $\square\omega \subset \zeta\omega$ so for each $\{\alpha_{\varsigma}\}$ in ω there correlates a sequence $\{\beta_{\varsigma}\}$ in ω and that is how that $\square\alpha_{\varsigma} = \zeta\beta_{\varsigma}$. As a consequence, $\lim_{\varsigma\to\infty} \zeta\beta_{\varsigma} = \lim_{\varsigma\to\infty} \square\alpha_{\varsigma} = \square \in \square\omega$. Thus we've got $\lim_{\varsigma\to\infty} \zeta\beta_{\varsigma} = \lim_{\varsigma\to\infty} \square\alpha_{\varsigma} = \lim_{\varsigma\to\infty} \square\alpha_{\varsigma} = \square$. Now probably $\square\,\omega$ is a subsetof ω which is closed, there is the possibility of a point $\rho \in \omega$ and that is how that $\square\rho = \square$. Now we're claiming

$\lim_{\varsigma\to\infty} \square\alpha_{\varsigma} = \square$. Now,

$$\square(\square\beta_{\varsigma}, \square\alpha_{\varsigma}) \leq \left\{ max \left\{ \begin{array}{c} \frac{\square(\zeta\beta_{\varsigma}, \square\alpha_{\varsigma})[\square(\zeta\beta_{\varsigma}, \square\beta_{\varsigma}) + \square(\square\alpha_{\varsigma}, \square\beta_{\varsigma})]}{\square(\square\alpha_{\varsigma}, \square\alpha_{\varsigma}) + \square(\square\alpha_{\varsigma}, \zeta\beta_{\varsigma})}, \\ \frac{\square(\zeta\alpha_{2\varsigma}, \square\beta_{\varsigma})[\square(\square\alpha_{\varsigma}, \square\alpha_{\varsigma}) + \square(\square\alpha_{\varsigma}, \zeta\beta_{\varsigma})]}{\square(\zeta\beta_{\varsigma}, \square\beta_{\varsigma}) + \square(\square\alpha_{\varsigma}, \square\beta_{\varsigma})} \end{array} \right\} \right\}^{\varphi}$$

Considering $\varsigma \to \infty$

$$\square(\square\beta_{\varsigma}, \square) \leq \left\{ max \left\{ \begin{array}{c} \frac{\square(\square, \square)[\square(\square, \square\beta_{\varsigma}) + \square(\square, \square\beta_{\varsigma})]}{\square(\square, \square) + \square(\square, \square)}, \\ \frac{\square(\square, \square\beta_{\varsigma})[\square(\square, \square) + \square(\square, \square)]}{\square(\square, \square\beta_{\varsigma}) + \square(\square, \square\beta_{\varsigma})} \end{array} \right\} \right\}^{\varphi}$$

$$= \left\{ max\{\square(\square, \square\beta_{\varsigma}), 1\} \right\}^{\varphi}$$

Which helps to ensure that $\lim_{\varsigma\to\infty} \square(\square\beta_{\varsigma}, \square) = 1$. Consequently

$$\lim_{\varsigma\to\infty} \square\beta_{\varsigma} = \lim_{\varsigma\to\infty} \square\alpha_{\varsigma} = \lim_{\varsigma\to\infty} \zeta\beta_{\varsigma} = \lim_{\varsigma\to\infty} \square\beta_{\varsigma} = \square = \square\rho$$

for an arbitrary $\rho \in \omega$. From the proof of Theorem 2.7 We can conveniently establish that □ isa invariant point which is common and unique as well of $\zeta, \square, \square$ and □. Also, one can easily establish that the pair $\zeta, \square$ Satisfies CLR_{ζ} property.

Similarly we can complete the proof for cases in which $\zeta\omega$ or $\square\omega$ or $\square\omega$ is a subset which is closed of ω. This is concluding the statement.

ARTICLE OUTCOME

Theorem *Suppose* $\zeta, \square, \square$ *and* □ *are functions of a multiplicative semi metric space* $(\omega, \square)$ *into itself satisfying the specifications*(□*1*); (□*2*), (□*3*), (□*5*) *and the following condition:*

(□7)*if the pair* □, □ *satisfiesReciprocally-continuous* (□□)*orWeakly reciprocally-continuous* (□□) *or S- Weakly reciprocally-continuous* (□□) *property. Then* □, □, □ *and* □ *have a invariant Point which is common and unique as well.*

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