On Mysterious Number 6174 ; Akaprekar Constant

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Abstract

The primary objective of this article is to chase the mysterybehind the fourth order Kaprekar constant 6174 using the principles of Linear Algebra and to hunt the other ordered Kaprekar constants. Kaprekarconstant 6174 has been an interesting one with its own mystery for decades in the research field of Analytical Number theory. Many recreational and experimental Mathematicians have been studying this number for years. In this research paper an attempt has been made to trace the mystery behindthatKaprekar constant 6174 and other Kaprekar constants of other orders. The method discussed here ensures a way to hunt Kaprekar constants of any order. Furthermore the proof of the existence and uniqueness of third order Kaprekar constant has been proposed and this proof can be implemented to find the Kaprekar constants of any order if they exist.

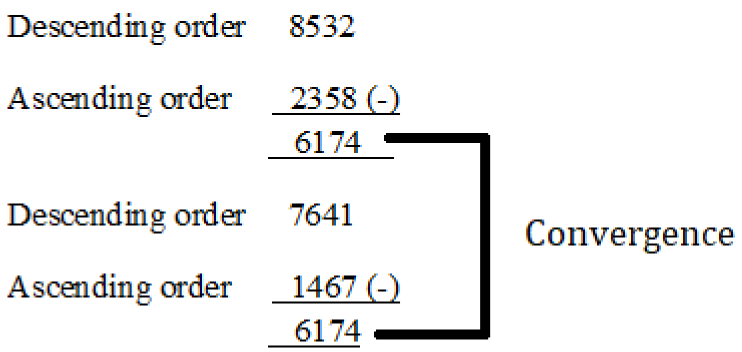
Keywords: Convergence, Kaprekar Process, Kaprekar Constant, Derrangements, NHLE, Bijection, Gauss-Jordan Method, Augmented Matrix, Rank, Echelon Matrix.

1. INTRODUCTION

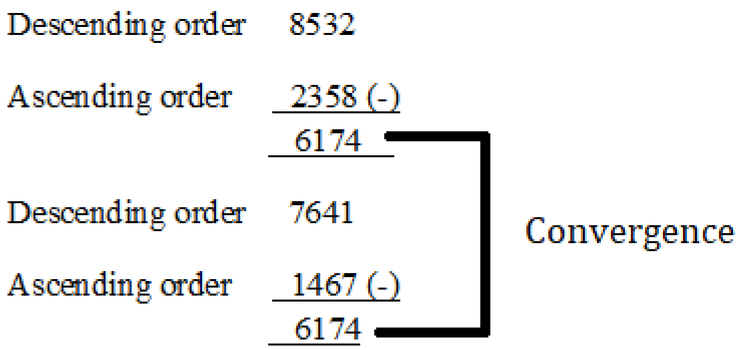
In 1947, DattatreyaRamadrandraKaprekar invented an astonishing property of the four digit number 6174 which was named as Kaprekar constant in the field of Recreational Mathematics. Take any four digits $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}. \boldsymbol{\delta}$ from the ten digits 0,1,2,…9 such that $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma} = \boldsymbol{\delta}$ is not satisfied. Arrange them in descending and ascending orders and make sure about these orders are having four digits. Now subtract latter from the former. Now one can be left with four digits. Repeat the process again. After a finite number of steps (at most 7 steps) the continuation of this process leaves a fixed four digit number 6174. Thus mystery was discovered by an Indian recreational mathematician Dr. Kaprekar and the constant 6174 is called Kaprekar constant of fourth order.

Example (1) Take 0,0,7,9
Descending order 9700
Ascending order 0079 (-)
9621

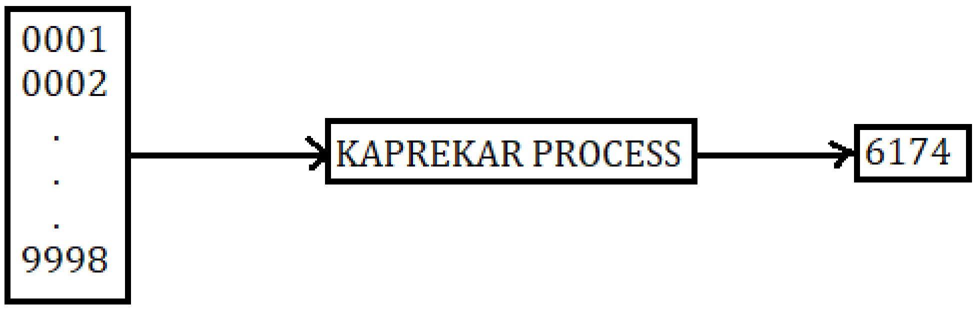
Descending order 9621
Ascending order 1269 (-)
8352



Example 2: Take 1,0,2,9
Descending order 9210
Ascending order 0129 (-)
9081
Descending order 9810
Ascending order 0189 (-)
9621
Descending order 9621
Ascending order 1269 (-)
8352.



Let S= {0,1,2,3,4,5,6,7,8,9}.
Let $\alpha, \beta, \gamma. \delta$ in decimal system be a four digit number constructing from the digits of S such that $\alpha = \beta = \gamma = \delta$ is not satisfied.
Let the collection of all these $\alpha\beta\gamma\delta$be K.
$\mathbf{K} = \{\alpha\beta\gamma\delta/\alpha, \beta, \gamma. \delta \; \square \; \mathbf{S \text{ and } \alpha = \beta = \gamma = \delta \text{ is not satisfied}}\}$
From the theory of combinatorics the order of K is $\mathbf{10^4 - 10 = 9990}$. If the Kaprekar process is applied to all these 9990 numbers then those processes converges to the mysterious number 6174 ofcourse it is an element of K



2. **DEVELOPING THE MATHEMATICAL ENVIRONMENT AROUND THE MYSTERY**

Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}. \boldsymbol{\delta}$ in decimal system bean arbitrary element of K. If it goes through Kaprekar's process after finite number of steps there will be four different digits say a,b,c,d with a>b>c>d>0.

Descending order	a	b	c	d
Ascending order	d	c	b	a .
	a-d	b-1-c	9+c-b	10+d-a.

Now the four digitedKaprekar constant exists if {a, b, c,d}={a-d, b-1-c, 9+c-b, 10+d-a}. This gives 4! = 24 systems of Non-Homogeneous Linear Equations(NHLE) in four unknowns. But a=a-d, b=b-1-c, c=9+c-b, and d=10+d-a are ruled out.So one should consider derrangements and they are $\mathbf{4}!\left(\frac{\mathbf{1}}{\mathbf{2}!}-\frac{\mathbf{1}}{\mathbf{3}!}+\frac{\mathbf{1}}{\mathbf{4}!}\right)=\mathbf{9}$ in number. These are enumerated as follows.

$a = c + 9 - b$

$b = a - d$

$c = d + 10 - a$ ....D_1

$d = b - 1 - c$

$a = b - 1 - c$

$b = a - d$

$c = d + 10 - a$ ....D_2

$d = c + 9 - b$

$a = d + 10 - a$

$b = c + 9 - b$

$c = b - 1 - c$ ....D_3

$d = a - d$

$a = b - 1 - c$

$b = c + 9 - b$

$d = a - d$ ....D_4

$c = d + 10 - a$

$a = c + 9 - b$

$b = d + 10 - a$

$c = a - d$ ....D_5

$d = b - 1 - c$

$a = d + 10 - a$

$b = a - d$

$c = b - 1 - c$ ....D_6

$d = c + 9 - b$

$a = b - 1 - c$

$b = d + 10 - a$

$c = a - d$ ....D_7

$d = c + 9 - b$

$a = c + 9 - b$

$c = b - 1 - c$

$b = d + 10 - a$ ....D_8

$d = a - d$

$a = d + 10 - a$

$b = c + 9 - b$

$c = a - d$ ....D_9

$d = c + 9 - b$

3. **CHASING THE MYSTERY BEHIND THE NUMBER 6174**

Consider the first derangement D_1

$a = c + 9 - b$

$b = a - d$

$c = d + 10 - a$ ....D_1

$d = b - 1 - c$

Expressing these into a system of nonhomogeneous linear equations in four unknowns a,b,c,d

$a + b - c = 9$

$a - b - d = 0$

$a + c - d = 10$

$b - c - d = 1$

Expressing this in matrix form AX=B one can get

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 10 \\ 1 \end{bmatrix}$$

One can solve this system by Gauss-Jordan method

Augmented matrix= $\begin{bmatrix} A & B \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & -1 & 0 & : & 9 \\ 1 & -1 & 0 & -1 & : & 0 \\ 1 & 0 & 1 & -1 & : & 10 \\ 0 & 1 & -1 & -1 & : & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 & : & 9 \\ 0 & -2 & 1 & -1 & : & -9 \\ 0 & -1 & 2 & -1 & : & 1 \\ 0 & 1 & -1 & -1 & : & 1 \end{bmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 & : & 9 \\ 0 & -2 & 1 & -1 & : & -9 \\ 0 & 0 & 3 & -1 & : & 11 \\ 0 & 0 & 0 & -10 & : & -10 \end{bmatrix} \quad 3R_4 + R_3$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 & : & 9 \\ 0 & -2 & 1 & -1 & : & -9 \\ 0 & 0 & 3 & -1 & : & 11 \\ 0 & 0 & 0 & 1 & : & 1 \end{bmatrix} \quad \frac{-1}{10}R_4 \quad ......(1)$$

$$A \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ an Echelon matrix

$$\rho([A:B]) = Number\ of\ nonzero\ rows = 4 = Rank\ of\ [A\ B]$$

Since $\rho(A) = \rho([A:B])$ = 4 = Number of unknowns, the system D1 is consistent and possesses unique solution.

From (1) one can observe that

d=1

3c-d=11 implies 3c-1=11

c=4

-2b+c-d=-9 implies -2b+4-1=-9

-2b+3=-9

b=6

a+b-c=9 implies a+6-4=9

a=7

Thus we have got a=7; b=6; c=4; d=1,

Descending order	7641	
Ascending order	1467 (-)	
	6174	Convergence
Descending order	7641	
Ascending order	1467 (-)	
	6174	

Hence the any arbitrary Kaprekar process of fourth order converges to the four digit number 6174 and is called the Kaprekar constant of fourth order.

Solving D1 tells that fourth order Kaprekar constant exists and the above observation tells the reason for the convergence of all fourth order Kaprekar routines and the point of convergence is 6174.

4. UNIQUENESS OF FOURTH ORDER KAPREKAR CONSTANT 6174

Take the second derrangement D2

$$\begin{aligned} a &= b - 1 - c \\ b &= a - d \\ c &= d + 10 - a \quad \quad ....D_2 \\ d &= c + 9 - b \end{aligned}$$

Expressing this into a system of nonhomogeneous linear equations in four unknowns, one can see

$a-b+c=-1$

$a-b-d=0$

$a+c-d=10$

$b-c+d=9$

In matrix form one can express this as

$$\begin{bmatrix}1 & -1 & 1 & 0\\ 1 & -1 & 0 & -1\\ 1 & 0 & 1 & -1\\ 0 & 1 & -1 & 1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}-1\\ 0\\ 10\\ 9\end{bmatrix}$$

In symbols it is written as AX=B

Augmented matrix is [A : B]

$$\sim\begin{bmatrix}1 & -1 & 1 & 0 & -1\\ 1 & -1 & 0 & -1 & 0\\ 1 & 0 & 1 & -1 & 10\\ 0 & 1 & -1 & 1 & 9\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & -1 & 1 & 0 & -1\\ 0 & 0 & -1 & -1 & 1\\ 0 & 1 & 0 & -1 & 11\\ 0 & 1 & -1 & 1 & 9\end{bmatrix}\qquad \begin{matrix}R_2-R_1\\ R_3-R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & -1 & 1 & 0 & -1\\ 0 & 0 & -1 & -1 & 1\\ 0 & 0 & 1 & -2 & 2\\ 0 & 1 & -1 & 1 & 9\end{bmatrix}\qquad R_3-R_4$$

$$\sim\begin{bmatrix}1 & -1 & 1 & 0 & -1\\ 0 & 1 & -1 & 1 & 9\\ 0 & 0 & -1 & -1 & 1\\ 0 & 0 & 1 & -2 & 2\end{bmatrix}\qquad \begin{matrix}R_2\to R_3\\ R_4\to R_2\\ R_3\to R_4\end{matrix}$$

$$\sim\begin{bmatrix}1 & -1 & 1 & 0 & -1\\ 0 & 1 & -1 & 1 & 9\\ 0 & 0 & -1 & -1 & 1\\ 0 & 0 & 0 & -3 & 3\end{bmatrix}\qquad R_4+R_3\to R_4$$

From this one can get -3d=3 $\Rightarrow$ d=-1

-c-d=1 $\Rightarrow$ -c+1=1 $\Rightarrow$ c=0

b-c+d=9 $\Rightarrow$ b=10

a-b+c=-1 $\Rightarrow$ a=9

As a,b,c,d are in {1,2,3,…9} this solution is not valid. Similarly remaining all seven derrangements fail to give a solution and consequently the uniqueness of the 4th order Kaprerkar constanthas been established.

5. **HUNTING FOR THE THIRD ORDER KAPREKAR CONSTANT**

Let **α β γ** in decimal system be an arbitrary element of K. If it goes through Kaprekar's process then after a finite number of steps there will three different digits a,b,c with a>b>c>0.

Descending order a b c
Ascending order c b a .
a-1-c 9 10+c-a
Now the three digit Kaprekar constant exists if {a, b, c} = {a-1-c, 9, 10+c-a}
This gives 3! = 6 systems of NHLE in three unknowns.
But a= a-1-c, b=9, c=10+c-a are impossible as a=a-1-c $\Rightarrow$ c=-1 (invalid)
Since b is less than a, b=9 isinadmissible
c=10+c-a givesa=10 (invalid)

$$\begin{matrix} a & b & c \\ a-1-c & 9 & 10+c-a \end{matrix}$$

Out of these 6 systems only few derrangements are considered and they are totally$\mathbf{3}!\left(\frac{\mathbf{1}}{\mathbf{2}!}-\frac{\mathbf{1}}{\mathbf{3}!}\right)=\mathbf{2}$ in number

$$\left.\begin{aligned} a &= 9 \\ b &= 10+c-a \\ c &= a-1-c \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots D_1$$

$$\left.\begin{aligned} a &= 10+c-a \\ b &= a-1-c \\ c &= 9 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots D_2$$

Since D2 contains c=9, D2 is neglectd. Now take D1 and put a=9 in third. Then one can get c=9-1-c $\Rightarrow$ c=4
Take second of D1 which is b=10+c-a $\Rightarrow$ b=10+4-9 $\Rightarrow$ b=5
As a,b,c are satisfying the required constraints third order Kaprekar constant exists and equals 495. From this discussion it is easy to see that third order Kaprekar constant 495 exists and is unique.

6. CONCLUSION AND FUTURE RESEARCH

In this talk the mystery behind the wonderful number 6174 known as Kaprekar’s constant has been extensively discussed with the help of rules in Matrix Algebra. Further more the existence and uniqueness of third order Kaprekar constant 495 has been established. In the context of future research one can implement these ideas in searching the Kaprekar constants of any order. Moreover by using the MATLAB and Python one can extract the Kaprekar constants of any order if they exist.

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