A Comparitive Study on Differentiation and Automorphisms of Algebras

B. Mahaboob[1], M. Rajaiah[2], C. Narayana[3], V.B.V.N. Prasad[4], K. Prasad [5], Y. Hari Krishna[6]
[1]*Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur A.P. India. E-mail: bmahaboob750@gmail.com*
[2]*Department of Mathematics, Audisankara College of Engineering & Technology (Autonomous), Gudur, SPSR Nellore, A.P India.*
E-mail: rajagopal1402@gmail.com
[3]*Department of Mathematics, Sri Harsha Institute of P.G Studies, Nellore, A.P.*
Email: nareva.nlr@gmail.com
[4]*Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur A.P. India. E-mail: vbvnprasad@kluniversity.in*
[5]*Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur, A.P. India.E-mail: prasad.qiscet@gmail.com*
[6]*Department of Mathematics, ANURAG Engineering College, Anathgiri (v), Kodad, Suryapet, Telangana, India.E-mail: yaraganihari@gmail.com*

Abstract-

This research article explores on metric, normal and division algebras and one of the objectives of this talk is to establish a fact that R, C and H are normal algebras .In this paper an innovative proof of a proposition namely "the metric algebra H^2 is a normed algebra", has been proposed. Besides a specific formula is evaluated for every differentiation of algebra. Furthermore an attempt has been made to present some fundamental characteristic properties of Cayley Algebra.The concepts namely doubling of algebras, metric algebras are discussed in an innovative way so that it shows a path to young researchers. They can have a glance on their future research by thinking over the phenomena namely structural constants of a Lie Algebra and the representation of the Lie Algebra by generators and relations.

Keywords: Metric Algebra, Normal Algebra, Automorphism, Doubled Algebra, Lie Algebra, Cayley Algebra.

1. Introduction:

V.V.Bavula[1] ,in 2005,in his research article ,proposed the inversion formula for automorphisms of the Weyl Algebras and Polynomial Algebras.In 2020,Bibinur Duisengalieva [2] et.al.,in their paper, investigated Tame and Wild automorphisms of differential polynomial algebras of rank 2.In 1979,Michihiko Mastuda[3],in his research article, discovered the concept of the group of automorphisms of a differential algebraic function field.In 1998,Helena Albuquerque[4] et.al.,in their research article,proposed a new approach to Octonions and Cayley Algebras.Cristina Flaut[5],in 2014,in his research article studied about some properties of algebras obtained by the Cayley-Dickson process

The construction of quaternions from C (the set of complex numbers) is just same as the construction of complex numbers from R (the set of real numbers). Suppose B(R) is any algebra where a conjunction $b \to \bar{b}$ is defined. Let B^2 be a vector space and it is the direct sumof B and itself.

$$B^2 = \{(b, c)/b, c \in B\}$$

Define multiplication in B^2 as

$$(b,c)(p,q)=(bp-\bar{q}c, c\bar{p}+qb)$$

The association $b \to (b,0)$ is a monomorphism of B into B^2.
b is identified with (b,0) and the algebra B is considered sub algebra of B^2. If B is algebra with unity 1, then (1,0) is identity of B^2. Suppose f=(0,1) and consequently cf=(0,c),(b,c)=b+cf.
From this it is clear that every element of B^2 is expressed as b+cf in one and only one way with the following properties

b(cf) = (bc)f
(bf)c = $(b\bar{c})$f
(bc)(cf) = $-\bar{c}$ b

Multiplication in B^2 is defined by adding distributive property to the above Clearly $f^2 = -1$.
For instance a doubling R^2 of R is the algebra C and the doubling C^2 of C is the algebra of quaternion H. The first one needs no proof. But to prove the second, complex numbers are to be expressed explicitly as η =b+cf of C^2.
In other words $b=b_0+b_1\alpha,\quad c=b_2+b_3\alpha,\ f=\beta,\ \alpha f=\gamma$
then η be come $b_0+b_1\alpha+b_2\beta+b_3\gamma$
For any b of B, fb=$\bar{b}$ f, the algebra B^2 like the quaternion algebra H in non-identity function. In the same way as b (cf) = (cb) f, the algebra B^2 will become non associative under the condition that B is non commutative..
In order to construct a doubling a conjugation in B^2 is defined as $\overline{b+cf}=\bar{b}-cf$
It is obvious that this association is linear as well as involutory. A simple observation depicts that it is anti-automorphism.

2. Metric, Normed and Division Algebras

A unit algebra B on R is called a METRIC ALGEBRA when a conjunction $b \to \bar{b}$ is defined over it and satisfies a property that

For any $b \in B, b\bar{b} \in R$ and is positive for b > 0.

A real number $|b|=\sqrt{b\bar{b}}$ is known as the NORM of b and $|b|=0 \Leftrightarrow b=0$ it is obvious that in an arbitrary METRIC ALGEBRA 'B' the relation $(m,n)=\frac{m\bar{n}+\bar{m}n}{2}$ defines a scalar product. Hence any arbitrary metric algebra is essentially a EUCLIDEAN SPACE with respect to the product. The norm of b means its length . The orthogonal compliment of the unity in a metric algebra B is given by B^1
$b \in$ B is written as $b=\mu+b'$ and $\mu \in R, b' \in B'$, $\bar{b}=\mu-b'$.

$$\bar{b}\in B' \Leftrightarrow \bar{b}=-b \text{ and } b\in R \Leftrightarrow \bar{b}=b$$

According to definition $m\bar{n}+\bar{m}n=2(m.n), \forall m,n \in B$ Moreover $nm=-mn \;\Leftrightarrow\; m \perp n\;, \forall m,n \in B'$

It is clear that if B is any metric algebra the algebra B^2 becomes metric.

In fact , $(b+cf)(\overline{b+cf})=(b+cf)(\bar{b}-cf)=b\bar{b}+c\bar{c}\;, \forall b+cf \in B^2$

A scalar product in B^2 is defined by $(b+cf, p+qf)=(b,p)+(c,q)$

Now B^2=B $\oplus$ B change in to Euclidean spaces sum.

Hence all algebras R,R^2=C, C^2=H, H^2,…are proved as metric algebra.

A Euclidean space finite dimensional algebra B is called a normal algebra if $|bc| = |b||c|$, $\forall b, c \in B$. In this for $b \neq 0$ the functions $m \to \frac{mb}{|b|}$ and $m \to \frac{bm}{|b|}$ are isometric so that one one onto mappings. Hence for arbitrary Chosen $c \in B$ the bm=C and mb=c have unique solution B. In this Light the normed linear algebra B becomes a division algebra R, C and H are some examples for normed Algebras. The metric algebra H^2 is also seen as normed by the following Hurwitz's results tells that R,C,H and H^2
are the only four normed algebras. If an automorphism $\psi : B \to B$ of a metric algebra is commutative with a conjugation, then it will become an orthogonal operator. In converse one can have if an automorphism $\psi : B \to B$ of a metric algebra B is orthogonal then as ψ 1=1 it maps itself the subspace B^1 and hence commutative with respect the conjuction. It is clear that if B is normed algebra then an arbitrary automorphism $\psi : B \to B$ is an orthogonal operator. To prove this it is enough if the following is shown, if $|b| = 1$ then $|\psi b| = 1$

But if $|\psi b| < 1$ then $|\psi b^l| = |(\psi b)^l| = |\psi b|^l \to 0$ as $l \to \infty$

That is $\psi b^l \to 0$ so that $b^l \to 0$

Hence $|b|^l = |b^l| \to 0$ and this is not possible for $|b| = 1$.

In this similar function if $|\psi b| > 1$ then $|b|^l \to \infty$ $|b^l| \to \infty$ as $l \to \infty$ and this is also non possible, so the conclusion becomes $|\psi b| = 1$

3. Specific formula of any differentiation of algebra

In each and every algebra B some basis is fixed as a rule Hence the group of orthogonal operators $B \to B$ are Identifies with the group G(t) of orthogonal matrices In the light of this for the group Aut (B) of automorphism of a normed linear algebra B set inclusion is defined as Aut B$\subset$G (t) in which t = dim B since every automorphism $\psi : B \to B$ is uniquely defined as $\psi' : B' \to B'$. ϕ and ψ is identified with ψ' that Aut B$\subset$ G(t-1)

Particularly, for n=2 this is a fact for B=C. As G(1)=Z_2 the group Aut C is a group order 2 with identity automorhism id and an automorhpism of a complex conjugate as $b \to \bar{b}$. In particularly Der C=0

In a straight forward way the expressions Aut C = Z_2 and Der C=0 are easily got. In fact being a linear operator as R which maps one to one any arbitrary automorphism $\psi : C \to C$ is uniquely defined by a number $\psi\alpha$ As $\alpha^2 = -1$, there should be an equation of type $(\psi\alpha)^2 = -1$ for that number so $\psi\alpha = 1$ or -1 and this presents an identity operator and complex conjugation. In the same manner any differentiation $d : C \to C$ is defined in only one way by a number d_i and it follows the following property $d\alpha.\alpha + \alpha.d\alpha = d(\alpha^2) = -d\alpha = 0$
this is possible when $d\alpha = 0$. These simple ideas give a generalization to the case of any arbitrary doubled algebra B^2 namely any differentiation d of an algebra B^2 is defined by $db = d_0 b + Eb.f, df = m_0 + n_0 f$
this linear operator d_0, $E : B \to B$ and the two elements $m_0, n_o \in B$ with the differentiation d are uniquely recovered from d_0, E and m_0, n_o in the light of the equation

$$d(b + cf) = db + dc.f + c.df$$

that is $d(b + cf) = (d_0 b - Ec + cm_o) + (Eb + d_0 c + n_0 c)f$ ....(1)

In order to depict the linear algebra Der B^2, it is enough to describe (d_0, E, m_0, n_0) for which the above relation gives the differentiation of the algebra B^2 . In order to get relations on d_0,E,m_0,n_0 ensuring the inclusion d $\in$ Der B^2 it is essential that in the expression

$$d\left(\eta\varepsilon\right)=d\eta.\varepsilon+\eta.d\varepsilon..............(2)$$

Here $\eta=b+cf\,,\varepsilon=m+nf\;\in B^2$

d is to be expression terms of (1) , all multiplications are to be traced out and the coefficients of f and 1 at the right and left are to be equated . For instance at $\eta=b,\varepsilon=y$ one can get the relation

$$d_0(bm)+E(by)f=(d_0b.m+b.d_0)+(Eb.\overline{m}+Ey.b)f$$

From this it is evident that d_0 is a differentiation of the algebra B and E can satisfy

$$E(by)=Eb.\overline{m}+Em.b..................(3)$$

The same relations are got at $\eta=cf$ and $\varepsilon=m$, at $\eta=b$, at $\eta=cf$ and $\varepsilon=nf$. It is appropriate to trace a GS of (3) and then these supplementary identities are considered. In addition to this it is useful think from the equations $m_0+\overline{m_0}=0$ and $n_0+\overline{n_0}=0$ which m turn give $\eta=\varepsilon=f\Rightarrow m_0,n_0\in B'$.

As Der C=0 one can see $d_0=0$ .To compute the operator E the fact $C=R^2$ is used and the same pressure is implemented by definitions the linear operators $X,Y:R\to R$ as

$Em=Xm+Ym\square i,m\in R\;\forall b,m\in R,$ (3) gives

$$X(bm)=Xb.m+Xm.b$$

$$Y(bm)=Yb.m+Ym.b$$

In other words X and Y are differentiation of the field R. So X=Y=0 that is Em=0 for $m\in R$.

Consequently $E(m+n\alpha)=E(n\alpha)=E\alpha.n$

That is $Eb=\delta_0$. Im b where $\delta_0=E\alpha$.

It is evident that (3) holds for all δ_0. This tells that any differentiation $d:H\to H$ should be of the form

$$d\left(b+c\beta\right)=\left(-\delta_0\operatorname{Im}c+Cm_0\right)+\left(\delta_0\operatorname{Im}b+n_0c\right)\beta\;.......(4)$$

Here $m_o,n_o,\delta_0\in C,m_o,n_o\in C'$, $m_0=\alpha b_0,n_0=\alpha c_0$, $b_0,c_0\in R$ .These expressions are substituted in (2) one can discovered that the function $d:H\to H$ given by (4) is a differentiation of the algebra H $\Leftrightarrow\operatorname{Re}\delta_0=-b_0$ i.e , $\delta_0=-b_0+id_0$ where $d_0\in R$.

This ensures any differentiation $d:H\to H$ of algebra H is obtained by the formula

$$d(b+C\beta)=\alpha(b_0\operatorname{Re}C-d_0\operatorname{Im}C)+[-(b_0\operatorname{Im}b+c_0\operatorname{Im}C)+\alpha(c_0\operatorname{Re}C+d_0\operatorname{Im}b)]....(5)$$

Here b_0,c_0,d_0 are reals.

The function (5) maps the subspace H^1 to it self and is given by the 3rd order matrix

$$\begin{pmatrix}0 & b_0 & -d_0\\ -b_0 & 0 & -c_0\\ d_0 & c_0 & 0\end{pmatrix}$$

In the basis of α,β,γ of that subspace.

Aut H coincides either with group HG (3) or with G(3).Consequently Aut H= HG(3).

It is obvious Aut H is connected………….(6)

Hence one can conclude that any automorphisms $H\to H$ is an internal automorphism of the form $\varepsilon\to\eta\varepsilon\eta^{-1}$. Group of internal automorphism $\varepsilon\to\eta\varepsilon\eta^{-1};n\in H^3$ of the algebra H Coincides with HG(3). Hence Aut H $\supset$ HG(3). The expression Aut H = G(3) is not possible

So Aut H = HG(3).

4. Cayley Algebra:

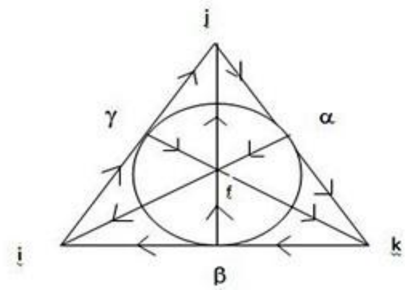
The algebra H^2 is called Cayley algebra whose elements are known as octaves are Cayley's numbers. This algebra is denoted by CA. According to definition each octaves can be put as

$\eta = b + cf$ where b and c are quaternion's and octaves are multiplied. The CA's basis posses 7 elements

$\alpha, \beta, \gamma, f, i = \alpha f, j = \beta f, k = \gamma f$................(7)

whose squares are equal to -1.

Pair wise product of these are represented by the following diagrammatically;



The product of any two of (7) equals up to a sign to the symbol on the same line segment or the circle and the sign is computed by the direction of the line segment. The algebra CA is not associative but alternative. If η and ε are any two elements then

$$(\eta\varepsilon)\varepsilon = \eta(\varepsilon\varepsilon), \eta(\eta\varepsilon) = (\eta\eta)\varepsilon$$

In fact, if $\eta = b + cf, \varepsilon = p + qf$ then

$$\eta\varepsilon = (bp - \bar{q}c) + (c\bar{p} + qb) f$$

$$(\eta\varepsilon)\varepsilon = [(bp - \bar{q}c)p - (c\bar{p} + qb)\bar{q}] + [\bar{p}(c\bar{p} + qb) + (bp - \bar{q}c)q] f$$

$$\varepsilon\varepsilon = (p^2 - \bar{q}q) + (q\bar{p} + qp) f$$

$$\eta(\varepsilon\varepsilon) = [b(p^2 - \bar{q}q) - (\overline{\bar{\bar{q}}p + pq})c] + [c(\overline{p^2 - \bar{q}q}) - (q\bar{p} + qp)b] f$$

As the $\bar{q}q = q\bar{q}$ and $p + \bar{p}$ are real numbers they commute with any quaternion and

$$b(p^2 - \bar{q}q) - (\overline{q\bar{p} + qp})b = bp^2 - b\bar{q}q - (p + \bar{p})\bar{q}c$$

$$= bp^2 - \bar{q}qb - \bar{q}c(p + \bar{p})$$

$$= (bp - \bar{q}c)p - \bar{q}(\overline{c\bar{p} + qb})$$

$$c(\overline{p^2 - \bar{q}q}) + \left(q\bar{p} + qp\right)b = c\bar{p}^2 - c\bar{q}q + q(p + \bar{p})b$$

$$= c\bar{p}^2 - \bar{q}qc + qb(p + \bar{p})$$

$$= (c\bar{p} + qb)\bar{p} + q(bp - \bar{q}c)$$

So $(\eta\varepsilon)\varepsilon = \eta(\varepsilon\varepsilon)$

In the similar fashion the following is established

$$\eta(\eta\varepsilon) = (\eta\eta)\varepsilon$$

The conjugate of CA is denoted by $\overline{b + cf} = \bar{b} - cf$

This can't change 1 and changes the sign of every element of (7). Hence elements of (7) will serve as a basis of a subspace CA[1]

5. Conclusions and Feature Research:

The above conversation throws a light on the concepts of doubling of algebras, metric algebras and normed algebras. An extensive discussion has been presented regarding the automorphisms and differentiations of metric algebras. In addition to this an analytical approach to the phenomenon namely differentiations of doubled algebra, differentiations and automorphisms of algebra has been given. In the context feature research enthusiastic researchers in this field of research can have a glance on some research ideas namely structural constants of the Lie algebra and the representation of the Lie algebra by generators and relations.

References

1). V. V. Bavula, The inversion formula for automorphisms of the Weyl algebras and polynomial algebras, J. Pure Appl. Algebra 210 (2007), 147-159; arXiv: math.RA/0512215.

2). Bibinur Duisengalieva (2020),"Tame and Wild automorphisms differential polynomial algebras of rank2", Fundamental and Applied Mathematics, 22(2019), No.4,101-114

3). Michihiko Matsuda, "The Group Of Automorphisms Of A Differential Algebraic Function Field", Nagoya Math. J. Vol. 74 (1979), 87-94

4). Helena Albuquerque, Shahn Majid(1998),"New approach to Octonions and Cayley Algebras",
arXiv: math/9810037[math,Q A]

5). Cristina Flaut (2014), "About some properties of algebras obtained by the Cayley-Dickson process", Palestine Journal of Mathematics, Vol. 3(Spec 1) (2014) , 388–394

6)Padmaja C.J.L., Bhagavan V.S., Srinivas B., Kameswari P.L.R. (2019), 'Enhancing the performance of rsa-type cryptosystems', International Journal of Scientific and Technology Research, 8(12), PP.1593-1598.

7)Ragamayi S., Bhargavi Y. (2019), 'A study of vague gamma-nearrings', International Journal of Scientific and Technology Research, 8(11), PP.3820-3823.

8)Ragamayi S., Bhargavi Y. (2019), 'Some results on homomorphism of vague ideal of a gamma-nearring', International Journal of Scientific and Technology Research, 8(11), PP.3809-3812.

9)Leelavathi R., Suresh Kumar G., Murty M.S.N., Srinivasa Rao R.V.N. (2019), 'Existence-uniqueness of solutions for fuzzy nabla initial value problems on time scales', Advances in Difference Equations, 2019(1), PP.-.

10)Leelavathi R., Suresh Kumar G. (2019), 'Characterization theorem for fuzzy functions on time scales under generalized nabla hukuhara difference', International Journal of Innovative Technology and Exploring Engineering, 8(8), PP.1704-1706.

11)Ram Prasad D., Kishore G.N.V., Srinuvasa Rao B., Adhi Lakshmi G. (2019), 'Results in partially ordered bipolar metric spacevia (α,ψ)contractive condition', International Journal of Recent Technology and Engineering, 8(1), PP.332-339.

12)Adhi Lakshmi G., Kishore G.N.V., Srinuvasa Rao B., Ram Prasad D. (2019), '(α,ψ) contractive results in ordered bipolar metric spaces', International Journal of Recent Technology and Engineering, 8(1), PP.319-324.

13)Ram Prasad D., Kishore G.N.V., Victor D.W.J., Srinuvasa Rao B. (2019), 'C*-algebra valued fuzzy soft metric space and related fixed point results by using triangular Î‘-admissible maps with application to nonlinear integral equations', International Journal of Recent Technology and Engineering, 8(1), PP.68-72.

14)Seetha Mani P., Sarala Y., Jaya Lalitha G. (2019), 'Primeradicals in ternary semi groups', International Journal of Innovative Technology and Exploring Engineering, 8(6 Special Issue 4), PP.1403-1404.

15)Amarendra Babu V., Ramya Swetha R., Anitha T. (2019), 'Vague seperation and connectedness', International Journal of Innovative Technology and Exploring Engineering, 8(6 Special Issue 4), PP.1414-1417.