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On Some Coefficient Inequalities For Certain Classes Of Regular And Multivalent Functions With Differential Operator

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Abstract

In this paper the author establishes the new result corresponds to the generalized differential operator $E_{\eta,s}^{\delta,m}\Im(z)$, ***related to multivalent regular functions. Also compute the Fekete szegö coefficient estimates are obtained for*** $\left|a_{s+2}-\lambda a_{s+1}^2\right|$ ***when*** $\lambda \geq 1,$ ***with sharpness for the operator, and also point out, as particular cases, the results obtained earlier by various authors.***

Keywords: Multivalent and regular functions, coefficient estimates, multivalent starlike and convex function, Fekete-Szegö problem.

1. INTRODUCTION

Let A_s be the category of functions $\Im(z)$ and the expansion

$$\Im(z) = z^s + \sum_{\tau=1}^{\infty} a_{\tau+s} z^{\tau+s} \quad (s \in N = \{1,2,3,...\}) \; (1)$$

which are holomorphic within the unit disk $\Theta = \{z : |z| < 1\}$ and let $A = A_1$. A function $\Im \in A_s$ observe by way of equ (1) is said to be multivalently starlike if

$$\Re\left(\frac{z\Im'(z)}{s\Im(z)}\right) > 0, \quad (z \in \Theta).$$

We denote this category of functions by means of S_s^*. Note that the category S_s^* dicrese to $S_1^* := S^*$, the category of starlike in Θ, introduced by Robertson [10].

A function $\Im \in A_s$ is multivalently convex if

$$\Re\frac{1}{s}\left(\frac{z\Im'(z) + z^2\Im''(z)}{z\Im'(z)}\right) > 0, \qquad (z \in \Theta).$$

We denote with the aid of C_s the familiar subcategory of A_s. In case $s = 1$, $C_1 := C$ the category of convex functions in Θ, brought by way of Robertson [10] (also see [2]).

A function $\Im(z)$ belonging to A_s is strongly starlike of order η in Θ, and denote with the aid of $SS^*(K)$ if

$$\left|\arg\left(\frac{z\Im'(z)}{\Im(z)}\right)\right| < \frac{\pi}{2}\gamma \;\; (0 < \gamma \leq s, \quad z \in \Theta). \; (2)$$

If $\Im(z) \in A_s$ satisfies

$$\left|\arg\left(1+\frac{z\Im'(z)}{\Im(z)}\right)\right| < \frac{\pi}{2}\gamma \quad (0<\gamma\le s, \quad z\in\Theta)\ (3)$$

Let $M_s(m,\delta,\eta,\gamma)$ be the category of regular, $\Im$ defined within the open unit disk Θ

$$\Re\left(\frac{E_{\eta,s}^{\delta,m+1}\Im(z)}{E_{\eta,s}^{\delta,m}g(z)}\right) > 0 \quad (\delta\in N, \delta, m, s\in N_0 = N\cup\{0\})$$

for some $g \in R_\gamma^s(m,\delta,\eta,\gamma)$.

The author define the following differential operator $E_{\eta,s}^{\delta,m} : A_s \to A_s$ by

$$E_{\eta,s}^{\delta,m}\Im(z) = z^s + \sum_{\tau=1}^{\infty}\Gamma_\tau(\tau+s)^m a_{\tau+s} z^{\tau+s}\ (4)$$

where $\Gamma_\tau = \dfrac{(\tau+s+\delta-1)!(\eta-1)!}{\delta!(\tau+s+\eta-2)!}$,

with the assist of the differential operator $E_{\eta,s}^{\delta,m}$, we say that

$$\left|\arg\left(\frac{E_{\eta,s}^{\delta,m+1}\Im(z)}{E_{\eta,s}^{\delta,m}\Im(z)}\right)\right| < \frac{\pi}{2}\gamma \quad (\delta\in N, \delta, m, s\in N_0 = N\cup\{0\})$$

for some $\gamma\ (0<\gamma\le s)$ and for all $z\in\Theta$.

Note that

$L_K^s(0,0,1,\gamma,1) = SS_s^*(K)$

and

$L_K^s(1,0,1,\gamma,1) = SC_s(K)$.

If $g(z)$ follow to A_s is said to be in the category $R_s(m,\delta,\eta,\gamma)$ denoted the category of regular function $g(z)$ by

$$\left|\arg\left(\frac{E_{\eta,s}^{\delta,m+1}g(z)}{E_{\eta,s}^{\delta,m}g(z)}\right)\right| < \frac{\pi}{2}\gamma \qquad (0\le\gamma\le s)$$

and for all $z\in\Theta$.

For the category S of multivalent regular function, [2] obtained the higher then point of $\left|a_{s+2}-\lambda a_{s+1}^2\right|$ when λ is non-imaginary. The upper bounded for $\left|a_{s+2}-\lambda a_{s+1}^2\right|$ is developed independent by many various authors. In this present work the author achive a sharp upper bounded for $\left|a_{s+2}-\lambda a_{s+1}^2\right|$ when $\lambda\ge 1$, $\Im(z)$ belongs to the category of functions as follows:

Definition 1:

Let $\Im(z)\in A_s$ *and then* $\Im\in M_s(m,\delta,\eta,\gamma)$ *iff there exist* $g\in R_{\tau,s}(m,\delta,\eta,\gamma)$ *so that*

$$R\left(\frac{E_{\eta,s}^{\delta,m+1}\Im(z)}{E_{\eta,s}^{\delta,m}g(z)}\right) > 0\ (5)$$

where $g(z) = z^s + b_{s+1}z^{s+1} + b_{s+2}z^{s+2} + \ldots$.

Note that $M_s(0,0,1,\gamma) = K_s(\gamma)$ the category of multivalent close – to - convex.

2. Main Result

Lemma 2 *Let* $\xi \in P$, ξ *be regular in* Θ *and follow by* $\xi(z) = 1 + \upsilon_1 z + \upsilon_2 z^2 + \ldots$ *and* $\Re(\xi(z)) > 0$ *for* $z \in \Theta$. *Then*

$$\left|\upsilon_2 - \frac{\upsilon_1^2}{2}\right| \le 2 - \frac{|\upsilon_1^2|}{2}. \quad (6)$$

Theorem 3 *Let* $\Im(z) \in \mathrm{M}_s(m,\delta,\eta,\gamma)$ *and observe by way of (1.1). Then for* $\gamma \ge 1$ *and* $\lambda \ge 1$ *we have that sharp disparity*

$$\left|a_{s+2} - \lambda a_{s+1}^2\right| \le 4\left[\frac{\gamma^2\left[\lambda\Gamma_2(s+2)^m - \frac{q}{2}\Gamma_1^2(s+1)^{2m+1}\right]}{s^2\Gamma_1^2\Gamma_2(s+1)^{2m+2}(s+2)^m}\right] + \frac{4s\lambda\Gamma_2(s+2)^{m+1}(1+2\gamma) - 2\Gamma_1^2(s+1)^{2m+2}(s+2\gamma)}{s\Gamma_1^2\Gamma_2(s+1)^{2m+2}(s+2)^{m+1}}.$$

(7)

Proof. Let $\Im(z) \in \mathrm{M}_s(m,\delta,\eta,\gamma)$ It observe by way of (1) that

$$\mathrm{E}_{\mu,s}^{\delta,m+1}\Im(z) = \mathrm{E}_{\mu,s}^{\delta,m} g(z)\varphi(z), \quad (8)$$

for $z \in \Theta$, with $\varphi \in \mathrm{P}$ observe by way of $\varphi(z) = 1 + \varphi_1 z + \varphi_2 z^2 + \varphi_3 z^3 + \ldots$. Equating coefficients, we obtain

$$\Gamma_1(s+1)^{m+1} a_{s+1} = \Gamma_1(s+1)^m b_{s+1} + \varphi_1 \quad (9)$$

and

$$\Gamma_2(s+2)^{m+2} a_{s+2} = \Gamma_2(s+2)^m b_{s+2} + \Gamma_1(s+1)^m b_{s+1}\varphi_1 + \varphi_2. \quad (10)$$

Also, it follows form equ (4) that

$$\mathrm{E}_{\mu,s}^{\delta,m+1} g(z) = \mathrm{E}_{\mu,s}^{\delta,m} g(z)\left[\Im(z)\right]^\gamma,$$

where for $z \in \Theta$, $h \in \mathrm{P}$ and

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \ldots.$$

Thus equating coefficients, we obtain

$$\Gamma_1 s(s+1)^m b_{s+1} = \gamma h_1, \quad (11)$$

and

$$\Gamma_2(s+1)(s+2)^m b_{s+1} = \gamma\left[h_2 + \left(\frac{\gamma}{s} + \frac{\gamma-1}{2}\right)h_1\right], \quad (12)$$

From equ (9), (10), (11) and (12) we have

$$a_{s+2} - \lambda a_{s+1}^2 = \frac{1}{\Gamma_2(s+2)^{m+1}}\left(\varphi_2 - \frac{1}{2}\varphi_1^2\right) + \frac{\gamma}{\Gamma_2(s+1)(s+2)^{m+1}}\left[h_2 - \frac{1}{2}h_1^2\right] + \left[\frac{\Gamma_1^2(s+1)^{2m+2} - 2\lambda\Gamma_2(s+2)^{m+1}}{2\Gamma_1^2\Gamma_2(s+1)^{2m+2}(s+2)^{m+1}}\right]\varphi_1^2 + \left[\frac{\gamma\Gamma_1^2(s+1)^{2m+2} - 2s\lambda\gamma\Gamma_2(s+2)^{m+1}}{s\Gamma_1^2\Gamma_2(s+1)^{2m+2}(s+2)^{m+1}}\right]\Im_1 h_1 \quad (13)$$
$$+\left[\frac{\gamma\left(\frac{\gamma}{s} + \frac{\gamma}{2}\right)\Gamma_1^2 s^2(s+1)^{2m+1} - \lambda\gamma^2\Gamma_2(s+2)^{m+1}}{s^2\Gamma_1^2\Gamma_2(s+1)^{2m+2}(s+2)^{m+1}}\right]h_1^2$$

Assume that $a_{s+2} - \lambda a_{s+1}^2$ is non-negative. Hence we now estimate $\Re\left(a_{s+2} - \lambda a_{s+1}^2\right)$, the above and by applying the Lemma 2 and let $h_1 = 2re^{i\theta}$, $\varphi_1 = 2Re^{i\phi}$, $r, R \in [0,1]$, $\theta, \phi \in [0, 2\pi]$, we obtain

$$(s+2)^{m+1}\Re\left[a_{s+2}-\lambda a_{s+1}^2\right]=\frac{1}{\Gamma_2}\Re\left(\varphi_2-\frac{1}{2}\varphi_1^2\right)$$

$$+\frac{\gamma}{\Gamma_2(s+1)}\Re\left[h_2-\frac{1}{2}h_1^2\right]$$

$$+\left[\frac{\gamma\Gamma_1^2(s+1)^{2m+2}-2s\lambda\gamma\Gamma_2(s+2)^{m+1}}{s\Gamma_1^2\Gamma_2(s+1)^{2m+2}}\right]\Re(\psi_1 h_1)+\left[\frac{\gamma\left(\frac{\gamma}{s}+\frac{\gamma}{2}\right)\Gamma_1^2 s^2(s+1)^{2m+1}-\lambda\gamma^2\Gamma_2(s+2)^{m+1}}{s^2\Gamma_1^2\Gamma_2(s+1)^{2m+2}}\right]\Re(h_1^2)\ (14)$$

$$\leq\frac{2}{\Gamma_2}\left(1-R^2\right)$$

$$+\left[\frac{\Gamma_1^2(s+1)^{2m+2}-2\lambda\Gamma_2(s+2)^{m+1}}{2\Gamma_1^2\Gamma_2(s+1)^{2m+2}}\right]4R^2\cos 2\phi\ +\frac{2\gamma}{\Gamma_2(s+1)}\left(1-r^2\right)$$

$$+\left[\frac{\gamma\Gamma_1^2(s+1)^{2m+2}-2s\lambda\gamma\Gamma_2(s+2)^{m+1}}{s\Gamma_1^2\Gamma_2(s+1)^{2m+2}}\right]$$

$$4rR\cos(\theta+\phi)$$

$$+\left[\frac{\gamma\left(\frac{\gamma}{s}+\frac{\gamma}{2}\right)\Gamma_1^2 s^2(s+1)^{2m+1}-\lambda\gamma^2\Gamma_2(s+2)^{m+1}}{s^2\Gamma_1^2\Gamma_2(s+1)^{2m+2}}\right]$$

$$4r^2\cos 2\theta$$

$$\leq\left[\frac{4\lambda\Gamma_2(s+2)^{m+1}-4\Gamma_1^2(s+1)^{2m+2}}{\Gamma_1^2\Gamma_2(s+1)^{2m+2}}\right]R^2+\left[\frac{4\gamma\left[2s\lambda\Gamma_2(s+2)^{m+1}-\Gamma_1^2(s+1)^{2m+2}\right]}{s\Gamma_1^2\Gamma_2(s+1)^{2m+2}}\right]rR$$

$$+\gamma\left[\frac{4\gamma\left[\lambda\Gamma_2(s+2)^{m+1}-\left(\frac{1}{s}+\frac{1}{2}\right)\Gamma_1^2 s^2(s+1)^{2m+1}\right]}{s^2\Gamma_1^2\Gamma_2(s+1)^{2m+2}}-\frac{2}{\Gamma_2(s+1)}\right]r^2+\frac{2}{\Gamma_2(s+1)}(\gamma+s+1)\ (15)$$

$$=\Psi(r,R).$$

Derivative partially above when $(\delta,\gamma,\lambda)\geq 1$ and $\eta\geq 0$, we study that $\Psi_{rr}\Psi_{RR}-(\Psi_{rR})^2<0$ therefore, the most of $\Psi(r,R)$ occurs on the limitations, hence the favored inequality follows by observing that

$$\Psi(r,R)\leq\Psi(1,1)$$

$$=4\left[\frac{\gamma^2(s+2)\left[\lambda\Gamma_2(s+2)^m-\frac{s}{2}\Gamma_1^2(s+1)^{2m+1}\right]}{s^2\Gamma_1^2\Gamma_2(s+1)^{2m+2}}\right]+\frac{4s\lambda\Gamma_2(s+2)^{m+1}(1+2\gamma)-2\Gamma_1^2(s+1)^{2m+2}(s+2\gamma)}{s\Gamma_1^2\Gamma_2(s+1)^{2m+2}}.$$

If (7) is attained when $h_1=\varphi_1=2i$ and $h_2=\varphi_2=-2$

Choosing $m=\delta=0$ and $\eta=1$ we get,

Corollary 4 *Let $\Im(z)\in \mathrm{L}(\gamma)$ and observe by way of (1). Then for $(\gamma,\lambda)\geq 1$ we have got the sharp disparity*

$$|a_{s+2}-\lambda a_{s+1}^{2}|\leq \frac{2\gamma^{2}\left[2\lambda-s(s+1)\right]}{s^{2}(s+1)^{2}} + \frac{4s\lambda(s+2)(2\gamma+1)-2(s+1)^{2}(2\gamma+s)}{s(s+1)^{2}(s+2)}. \quad (16)$$

Letting $m=\eta=1$ and $\delta=0$ we get,

Corollary 5 *Let* $\Im(z)\in L(1,0,1,\gamma)$ *and observe by way of (1). Then for* $(\gamma,\lambda)\geq 1$ *we have got the sharp disparity*

$$|a_{s+2}-\lambda a_{s+1}^{2}|\leq \frac{1}{s^{2}(s+1)^{4}(s+2)^{2}}\left[\begin{array}{l}2(s+2)\gamma^{2}\left[2(s+2)\lambda-s(s+1)^{2}\right]\\ +4s^{2}\lambda(s+2)^{2}(1+2\gamma)-2s(s+1)^{4}(s+2\gamma)\end{array}\right]. \quad (17)$$

Choosen $s=1$ we get following

Corollary 6[1] *Let* $\Im(z)\in L(m,\delta,\eta,\gamma)$ *and observe by way of (1). Then for* $\gamma\geq 1$ *and* $\lambda\geq 1$ *we have got the sharp disparity*

$$|a_{3}-\lambda a_{2}^{2}|\leq \frac{\gamma^{2}\left[3^{m}\lambda\eta^{2}(\delta+2)-2^{2m}\eta(\eta+1)(\delta+1)\right]}{2^{2m}3^{m}(\delta+2)(\delta+1)^{2}} + \frac{3^{m+1}\lambda^{2}(\delta+2)-2^{2m+1}\eta(\eta+1)(\delta+1)](2\gamma+1)}{2^{2m}3^{m+1}(\delta+2)(\delta+1)^{2}}. \quad (18)$$

Corollary 7[1] *Let* $\Im(z)\in K(\gamma)$ *and observe by way of (1). Then for* $(\gamma,\lambda)\geq 1$ *we have got the sharp disparity*

$$|a_{3}-\lambda a_{2}^{2}|\leq \gamma^{2}(\lambda-1)+\frac{(2\gamma+1)(3\lambda-2)}{3} \quad (19)$$

Corollary 8[1] *Let* $\Im(z)\in L(1,0,1,\gamma)$ *and observe by way of (1). Then for* $\gamma\geq 1$ *and* $\lambda\geq 1$ *we have got the sharp disparity*

$$|a_{3}-\lambda a_{2}^{2}|\leq \frac{1}{36}\left[3\gamma^{2}(3\lambda-4)+(9\lambda-8)(2\gamma+1)\right].$$

(20)

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