Meshfree Technique For The Solution Of Sine-Gordan Equation

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Abstract : In this manuscript, a meshless technique based on radial basis function is used for solving the Sine-Gordan equation. Radial basis function combined with pseudospectral approach is used to solve the nonlinear Sine-Gordan equation. The given equation is transformed to a system of ODEs with the help of radial basis function then an ODE solver is used to solve the transformed ODEs. Numerical example is established to showcase the validness and efficiency of the method. Error norms are calculated to check the accuracy. The obtained numerical results are in good accord with the exact solution existed in literature.

Keywords: Radial basis function, Collocation, Pseudospectral approach

1. Introduction

Various nonlinear phenomena arising in different fields are modelled with the support of non-linear partial differential equations (PDEs). Most of the nonlinear equations based upon real life problems are difficult to solve analytically. Numerical methods can be treated as an important tool in solving these nonlinear PDEs due to ease of implementation as compared to analytical approaches. In this paper, a numerical scheme is establish for nonlinear hyperbolic partial differential equation known as sine-Gordan equation having applications in various fields like differential geometry, realistic field theory and dislocation of crystals.

In this article, we consider the sine-Gordan equation which is given by:

$$u_{tt} = u_{xx} - sinu, \quad x \in (L_0, L_1), t > 0 \tag{1}$$

The initial conditions can be taken as

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x) \tag{2}$$

and the boundary conditions as

$$u(L_0,t) = f_0(t) \quad \text{and} \quad u(L_1,t) = f_1(t) \tag{3}$$

Numerically, the sine-Gordan equation has been analysed by various researchers by applying various numerical techniques. Dehghan and Shokri [1] solved the sine-Gordan equation with the help of collocation method. The modified decomposition method is applied by Kaya [2] to solve the sine-Gordan method. Hashim et al. [3] applied the variational iteration method to solve the equation. A fourth order numerical technique is applied by Bratsos [4] to find the numerical solution of sine-Gordan equation. Rashidinia and Mohammadi [5] applied a tensor spline technique to solve the equation.

Over the years various well known numerical methods are developed to solve PDEs such as finite difference, finite element, finite volume, variation iteration method. Almost all traditional numerical techniques require mesh discretization to solve PDEs sometime which is not possible due to the complex domain. This leads to the development of meshfree techniques. Meshfree methods are independent of the generation of mesh. Radial Basis Function (RBF) methods are such meshfree techniques which are very popular due to their meshless nature and ease of implementation. Fasshauer [6-8] established a new approach studied as radial basis function based pseudospectral method (RBF-PS). In RBF-PS method, RBF is combined with the highly accurate pseudospectral approach. In this paper, the one dimensional sine-Gordan equation is solved by RBF-PS method [9-12]. The numerical results obtained by solving the sine-Gordan equation is compared with the exact solution.

2. The governing equations and the method with radial basis functions:

Let us transformed the given sine-Gordan (1) as coupled equation by putting

$u_t = v\,,\quad v_t = u_{xx} - sinu,\, x \in (L_0, L_1), t > 0$ (4)

The given domain $[L_0, L_1]$ is divided into nodes $x_k = 1,2,3, \ldots N$. The RBF approximation for $u(x,t)$ and $v(x,t)$ can be written in the form as

$u_N = \sum_{k=1}^{N} \zeta^1{}_k\, \phi_k(\|x - x_k\|)$, $v_N = \sum_{k=1}^{N} \zeta^2{}_k\, \phi_k(\|x - x_k\|)$ (5)

where $\phi_k = \phi(r)\; and\; r = \|x - x_k\|$ denotes the Euclidean distance between the points x and x_k. as ϕ_k is the radial basis function.

Equation (5) evaluated at various nodes$x_k = 1,2,3, \ldots N$, we get

$u_N(x_i) = \sum_{k=1}^{N} \zeta^1{}_k\, \phi_k(\|x_i - x_k\|)$, $v_N(x_i) = \sum_{k=1}^{N} \zeta^2{}_k\, \phi_k(\|x_i - x_k\|)$ (6)

In the matrix form, we can write equation (3) as

$$U = AC_1, \quad V = AC_2 \qquad (7)$$

where $A_{ik} = \phi_k(\|x_i - x_k\|)$ are the radial basis functions at nodes and $C_1 = [\zeta^1{}_1, \zeta^1{}_2, \ldots, \zeta^1{}_N]^T$ and $C_2 = [\zeta^2{}_1, \zeta^2{}_2, \ldots, \zeta^2{}_N]^T$are the unknown interpolation coefficients.

Now, the derivative of u_N of (5) by differentiating the basis functions, as

$\frac{d}{dx_i} u_N(x_i) = \sum_{k=1}^{N} \zeta^1{}_k\, \frac{d}{dx_i} \phi_k(\|x_i - x_k\|)$ (8)

Again, evaluate (8) at the grid points, $x_k = 1,2,3, \ldots N$, we get

$U_x = A_x C_1$ (9)

where the entries of the derivative matrix A_x are $\frac{d}{dx} \phi_k(\|x_i - x_k\|)$.

The condition that the evaluation matrix A in (9) is invertible, depends on various factors like RBF selected and the chosen grid points. The matrix generated by using a positive definite RBFs is always lead to a non-singular matric and hence invertible.

Since A is invertible so from equation (9) $C_1 = A^{-1}U$ and equation (6) becomes

$U_x = A_x A^{-1} U = D_x U$ (10)

where $D_x$is known as Differentiation matrix

Similarly, one can find the differentiation matrix concerning the second and higher order derivatives, i.e

$U_{xx} = A_{xx} A^{-1} U = D_{xx} U$ (11)

Similarly $V_x = D_x V$ and $V_{xx} = D_{xx} V$

Using the approximations the coupled equation (4) can be written as

$\frac{dU_N}{dt} = V_N \qquad \frac{dV_N}{dt} = D_{xx} U_N - sinU_N$ (12)

By RBF-PS scheme, equation (1) reduces to a system of ODEs. The obtained ODEs can be discretized in time using any ODE solver like ode113, ode45 from MATLAB. We have used ode45 ODE solver to solve the resultant ODEs.

3. Numerical Results:

In this section, the RBF-PS method is applied for the numerical simulation of given equation. The accuracy of the proposed method is established with the help of L_2 and L_∞ error

$$L_2 \text{ error} = \sqrt{h \sum_{k=1}^{N} (U(x_k, t_n) - U_N(x_k, t_n))^2}$$

$$L_\infty \text{error} = \max_{1\le k\le N} |U(x_k, t_n) - U_N(x_k, t_n)|$$

for $0 \le t_n \le T$, where U and U_N denotes the exact and the numerical solutions. In this paper, we used a positive definite Cubic Matérn RBF given by $\varphi(r) = (15 + 15\epsilon r + 6(\epsilon r)^2 + (\epsilon r)^3)e^{-\epsilon r}$ as a basis function for approximation.

Example 1: Consider the sine-Gordan equation (1) over the domain [-3, 3]
The exact solution of the equation is given by

$$u(x,t) = 4\arctan\big(\exp\big(a(x - ct)\big)\big)$$

The initial conditions are

$$u(x,0) = 4\arctan(\exp(ax))$$

$u_t(x,0) = \frac{-4caexp(ax)}{1+\exp(2ax)}$,
where $a = 1/\sqrt{1-c^2}$ and c =0.5 represents the velocity of the wave.
The boundary conditions are extracted from the exact solution.
Table 1 represents various error norms for different values of t. Error norms are calculated with N=61 and Δt=0.001. The efficiency and accuracy of the method is checked by calculating the absolute error for different time levels. In Table 2, the obtained results are compared with exact solution. Figure 1 represents the performance of the solution graphically with Δt=0.001, N=61 for t = 1.

Table 1. *Error norm with shape parameter 0.208417 and* Δt=0.001*at some values of t.*

Time (t)	**0.25**	**0.50**	**0.75**	**1.0**
L_∞	4.3717E-05	3.6714e-04	1.600E-03	4.300E-03
L_2	3.3614E-07	2.3831e-06	1.2054E-05	3.6698E-05
RMS	2.7900E-05	8.4958e-05	3.4581E-04	3.9099E-08

Table 2. *Absolute error is shown for different values of x at t=0.001with N=61*

x		***t = 0.01***		***t = 0.1***
-2.5		*2.33E-07*		*7.89E-06*
-2		*4.06E-07*		*1.38E-05*
-1.5		*6.80E-07*		*2.32E-05*
-1		*9.94E-07*		*3.45E-05*
0		*9.48E-07*		*3.52E-05*
1		*1.30E-08*		*4.94E-06*
1.5		*9.38E-07*		*3.13E-05*
2		*9.99E-07*		*3.65E-05*
2.5		*6.87E-07*		*2.58E-05*

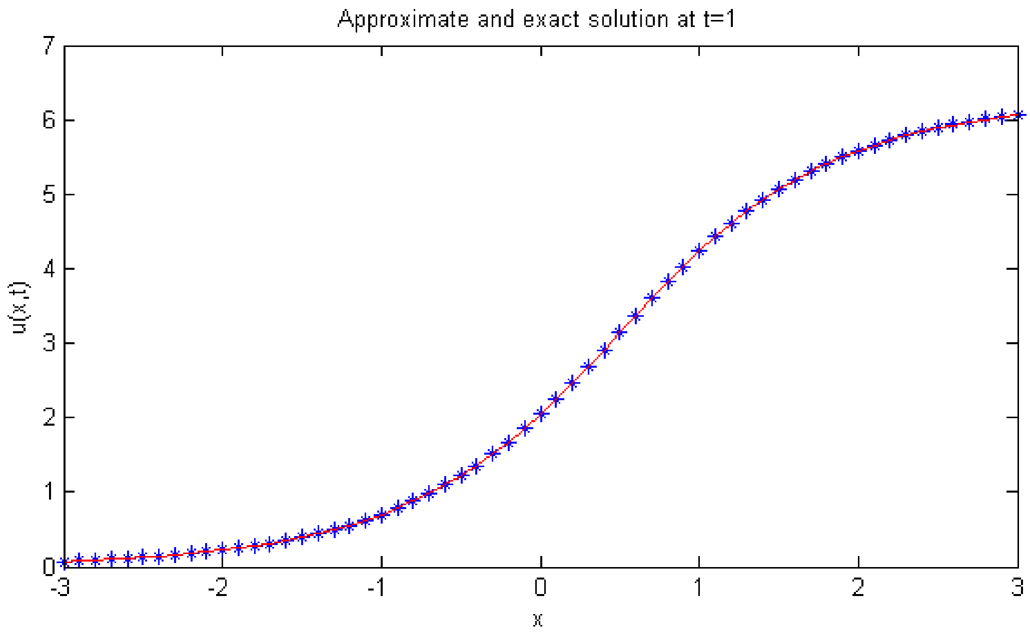

Figure 1: *Approximate and exact solution at t=1 with Δt=0.001, N=61*

4. Conclusion

Sine-Gordan equation is solved numerically with the help of radial basis function pseudospectral method. RBF-PS is easy to implement and simple. The applied method transforms the given equation into a system of ODEs. An ODE solver in MATLAB is used to solve the resultant ODEs. We use ODE45 to solve the ODEs. The obtained numerical solution are in good coordination with exact solution especially for the small value of t.

5. References

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