European Journal of Molecular & Clinical Medicine
ISSN 2515-8260 Volume 07, Issue 07, 2020

STUDY OF SOLUTION SWIFT-HOHENBERG EQUATION

Geeta Arora

Department of Mathematics, Lovely Professional University, Phagwara, Punjab, India
Email: geetadma@gmail.com

Abstract: The non-linear Swift-Hohenberg (SH) equation broadly exists as a model in the study of pattern formation. This equation is in focus of researchers because of the patterns existing in the solution. This equation models many of the fluids dynamics phenomenon and the genetic materials such as tissues in study of the pattern formation. Many of the researchers have obtained the solution of the equation based on different analytical and numerical approaches. This manuscript is dealt with the numerical solution of SH equation using the collocation approach using B-spline basis functions. For the different values of the parameters involved, the equation is studied graphically for the formation of patterns.
Keywords: Collocation Method; Swift-Hohenberg equation; fourth-order semi-linear equation; quasilinearization.

1. INTRODUCTION

Swift-Hohenberg (SH) equation is a fourth-order semi-linear equation studied by Swift and Hohenberg [1] in connection with Rayleigh–Bénard convection. This equation exists as a mathematical model to study fluid flow [2, 3] and in the study of lasers [4]. It is a expansion of the Fisher-Kolmogorov model which is . use to explore the pattern formation. The Swift-Hohenberg equation is extensively studied for different issues in pattern formation [5] of phenomenon in fluid dynamics, neural tissues and in phase field models [6].

Solution for SH equation has been studied for its periodicity and pattern formation by Carriaoa et al. [7]. The behavior of the equation with the existence and nonexistence of the Swift-Hohenberg equation are given by Chaparova et al. [8, 9]. Ogawa and Okuda [10] have studied the Swift-Hohenberg equation with perturbed boundary conditions. The behavior of solutions of the Swift–Hohenberg equation at large-time has been studied by Peletier and Rottschäfer [11]. The existence of homoclinic solutions for the Swift–Hohenberg model along with the equation for suspension bridge has been established by Smets and Berg [12]. The equation has been solved by Talay Akyildiz et al. [13] by applying homotopy analysis method (HAM). Crank-Nicolson-Adams-Bashforth scheme is used for the analysis of exact solution of equation by Kudryasha and Ryabay [14]. Dynamics of the stationary SH equation using piecewise Hermite basis functions is studied by Kumar et. al.[15]. Recently the equation has been solved using rational spline approach and nonstandard finite difference method by Zahra [16].

In this paper, the Swift-Hohenberg equation is solved numerically to establish the pattern formation solutions with periodicity using the proposed method computationally, followed by a brief description of the obtained results. The stability of the method is also discussed.

2. Description of Method

To obtain the numerical solution of the equation we have implemented the collocation method to the equation. In the collocation approach, a number of choices have to be made to approximate the solution of differential equations.

The method begins with a proper choice of basis functions $\{\phi_1, \phi_2, \ldots, \phi_N\}$ and a set points in the domain $[a,b]$. The domain $[a,b]$ is discretized as $a = x_0 < x_1, \ldots, x_{N-1} < x_N = b$ by considering a regular partition of the solution domain by the knots x_m with uniform step-length $h = x_{m+1} - x_m$ where, $m = 0, \ldots, N-1$.

The approximate solution can be written in form

$$U = \sum_{i=1}^{N} \alpha_i \phi_i(x) \quad (1)$$

Here, α_i are constants when the equation is a boundary value problem. However in the case of initial value problem α_i are the time-dependent coefficients.

In this paper, solution to the SH equation is obtained using collocation approach with quintic B-spline basis. Many authors have implemented the quintic B-splines basis functions in the collocation approach to solve differential equations both in linear and non linear forms. For instance, numerical solution of the Burgers' equation has been obtained by Sepehrian and Lashani [17] using quintic B-spline collocation method. Quintic B-spline Galerkin method was implemented to obtain the solutions of the RLW equation by Dag et al. [18]. The Korteweg–de Vries Burgers' (KdVB) and Korteweg–de Vries (KdV) equation is solved numerically by using quintic B-spline functions with finite elements by [19] and [20] respectively. Numerical Solution of the Kuramoto-Sivashinsky Equation [21] and Extended Fisher-Kolmogorov Equation [22] has been obtained by Mittal and Arora using quintic B-spline collocation method.

In this paper we have implemented the collocation method using quintic B-spline basis functions. To find solution of a partial differential equation using collocation method with B-spline basis function $B(x)$, the approximate solution $U(x,t)$ is taken as a linear combination of basis functions as follows

$$U(x,t) = \sum_{j=m-k+2}^{m+k-2} c_j(t) B_j(x) \quad (2)$$

where, k is the degree of the B-spline, m is the number of nodes and c_j are the time dependent parameters to be extracted using the boundary conditions. This approximate formula is used to obtain approximate numerical solutions of partial differential equations by using the approach given by Prenter [23].

3. Quintic B-spline Basis Function

The definition for the quintic B-spline basis function is given as

$$B_m(x)=\frac{1}{h^5}\begin{cases}(x-x_{m-3})^5 & x\in[x_{m-3},x_{m-2})\\ (x-x_{m-3})^5-6(x-x_{m-2})^5 & x\in[x_{m-2},x_{m-1})\\ (x-x_{m-3})^5-6(x-x_{m-2})^5+15(x-x_{m-1})^5 & x\in[x_{m-1},x_m)\\ (x_{m+3}-x)^5-6(x_{m+2}-x)^5+15(x_{m+1}-x)^5 & x\in[x_m,x_{m+1})\\ (x_{m+3}-x)^5-6(x_{m+2}-x)^5 & x\in[x_{m+1},x_{m+2})\\ (x_{m+3}-x)^5 & x\in[x_{m+2},x_{m+3})\\ 0 & otherwise\end{cases} \quad (3)$$

From the definition of B-spline basis functions, the values of $B_m(x)$ at different values of nodes can be obtained. Using definition of quintic B-spline (3) the values of $B_m(x)$ and its four derivatives can be calculated which are tabulated in Table 1.

On substituting $k=5$ in (2) and representing the approximate solution of the equation by $U(x,t)$, the approximate solution can be written as

$$U(x_m,t)=\sum_{j=m-3}^{m+3}c_j(t)B_j(x_m) \quad (4)$$

The approximate solution and its derivatives obtained by using the definition of the quintic B-spline basis functions can be written as

$$\begin{aligned} U(x_m,t) &= c_{m-2}+26c_{m-1}+66c_m+26c_{m+1}+c_{m+2}\\ hU'(x_m,t) &= 5(-c_{m-2}-10c_{m-1}+10c_{m+1}+c_{m+2})\\ h^2U''(x_m,t) &= 20(c_{m-2}+2c_{m-1}-6c_m+2c_{m+1}+c_{m+2})\\ h^3U'''(x_m,t) &= 60(-c_{m-2}+2c_{m-1}-2c_{m+1}+c_{m+2})\\ h^4U^{iv}(x_m,t) &= 120(c_{m-2}-4c_{m-1}+6c_m-4c_{m+1}+c_{m+2}) \end{aligned} \quad (5)$$

4. IMPLEMENTATION OF METHOD

Consider the Swift-Hohenberg equation

$$u_t+u_{xxxx}+2u_{xx}+u^3+(1-\alpha)u=0,\ x\in[a,b],\ t\in[0,T], \quad (6)$$

where, $\alpha\le 1$ is a parameter.

The approximate solution of the equation is obtained subject to boundary conditions

$$u(a,t)=g_0,\ \ u(b,t)=g_1, \quad (7)$$

$$u_{xx}(a,t)=0,\ \ u_{xx}(b,t)=0, \quad (8)$$

and initial condition

$$u(x,0)=f(x). \quad (9)$$

To implement the approach, the time derivative is discretized using finite difference with the Crank-Nicolson scheme applied to equation (6), to obtain

$$\left[\frac{u^{n+1}-u^n}{\Delta t}\right]+\left[\frac{(u_{xxxx})^{n+1}+(u_{xxxx})^n}{2}\right]+2\left[\frac{(u_{xx})^{n+1}+(u_{xx})^n}{2}\right]+\left[\frac{(u^3)^{n+1}+(u^3)^n}{2}\right]+(1-\alpha)\left[\frac{u^{n+1}+u^n}{2}\right]=0,$$

(10)

where Δt is the time-step.

The nonlinear term $(u^3)^{n+1}$ in the equation has been linearized using the quasilinearization formula. The linear equation is obtained by using the first and second terms of the Taylor's series expansion of the nonlinear term. The formula is obtained by getting (n+1)th iteration value by taking Taylor's Series about known nth iteration value as follows

$$X^{n+1}(u) = X^n(u) + (u^{n+1} - u^n)\left.\frac{\partial X}{\partial u}\right|^n \quad \text{where } n = 0,1,2\ldots \tag{11}$$

Here, $X(u)$ represents a nonlinear function of u.

Using the quasilinearization formula the nonlinear term $(u^3)^{n+1}$ can be written as

$$(u^3)^{n+1} = 3(u^2)^n u^{n+1} - 2(u^3)^n \tag{12}$$

Substituting the linearized expression (12) in equation (10) and simplifying we get

$$\begin{aligned} u^{n+1} + \frac{\Delta t}{2}\left[(u_{xxxx})^{n+1} + 2(u_{xx})^{n+1} + 3(u^2)^n u^{n+1} + (1-\alpha)u^{n+1}\right] = \\ u^n - \frac{\Delta t}{2}\left[(u_{xxxx})^n + 2(u_{xx})^n - (u^3)^n + (1-\alpha)u^n\right] \end{aligned} \tag{13}$$

Equation (13) may be written as

$$u^{n+1}\left(1 + \frac{\Delta t}{2}\left[3(u^2)^n + (1-\alpha)\right]\right) + u_{xx}^{n+1}\left(\frac{2\Delta t}{2}\right) + u_{xxxx}^{n+1}\left(\frac{\Delta t}{2}\right) = R \tag{14}$$

where, R is the right-hand side of equation (13).

Substituting the approximate solution for $u(x,t)$ and its derivatives given by (5), equation (14) can now be written in terms of unknown time parameters c_m's, which on simplification becomes

$$\begin{aligned} c_{m-2}^{n+1}(w+y+z) + c_{m-1}^{n+1}(26w+2y-4z) + c_m^{n+1}(66w-6y+6z) \\ + c_{m+1}^{n+1}(26w+2y-4z) + c_{m+2}^{n+1}(w+y+z) = R. \end{aligned} \tag{15}$$

where, $m = 0 \ to \ N.$

Here $c_m^n = c_m(t_n)$ and $t_n = n\Delta t$.

$$w = \left(1 + \frac{\Delta t}{2}\left[3(u^2)^n + (1-\alpha)\right]\right), \quad y = \frac{20\Delta t}{h^2}, \quad z = \frac{60\Delta t}{h^4}.$$

Equation (15) can be written in simplified form as

$$a_1 c_{m-2}^{n+1} + a_2 c_{m-1}^{n+1} + a_3 c_m^{n+1} + a_2 c_{m+1}^{n+1} + a_1 c_{m+2}^{n+1} = R \tag{16}$$

where, $m = 0 \ to \ N.$

For simplification we have defined

$a_1 = w+y+z, \qquad a_2 = 26w+2y-4z, \qquad a_3 = 66w-6y+6z,$

System (16) consists of $(N+1)$ linear equations in $(N+5)$ unknowns. The unknowns can be written as $(c_{-2}, c_{-1}, c_0, \ldots, c_N, c_{N+1}, c_{N+2})$

To obtain a unique solution of the obtained system four additional constraints c_{-1}, c_{-2}, c_{N+1}, and c_{N+2} are required. Imposition of the boundary conditions enables us to eliminate these parameters from the system. Eliminating c_{-1}, c_{-2}, c_{N+1} and c_{N+2} the system (16) get reduced to a penta-diagonal system of $(N+1)$ linear equations with $(N+1)$ unknowns, given by

$$AC_N = D_N$$

Here, $C_N = (c_0, c_1, c_2, \ldots, c_{N-1}, c_N)^T$, is the column matrix of unknown parameters to be evaluated, A is the coefficient matrix given by

$$A = \begin{bmatrix} 12a_1 - 3a_2 + a_3 & 0 & 0 & 0 & . & \cdots & 0 \\ a_2 - 3a_1 & a_3 - a_1 & a_2 & a_1 & 0 & 0 & . \\ a_1 & a_2 & a_3 & a_2 & a_1 & 0 & . \\ 0 & a_1 & a_2 & a_3 & a_2 & a_1 & \vdots \\ . & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & . & a_1 & a_2 & a_3 & a_2 & a_1 \\ . & . & 0 & a_1 & a_2 & a_3 - a_1 & a_2 - 3a_1 \\ 0 & . & \cdots & 0 & 0 & 0 & 12a_1 - 3a_2 + a_3 \end{bmatrix} \quad (17)$$

and

$$D_N = \left[R + \frac{g_0}{24}(2a_1 - a_2), \quad R - \frac{g_0}{24}a_1, \quad R \quad \ldots \quad \ldots \quad R, \quad R - \frac{g_1}{24}a_1, \quad R + \frac{g_1}{24}(2a_1 - a_2) \right]^T$$

Here R denotes the rhs of equation (13).
The resulted penta-diagonal system can be solved by a modified form of Thomas algorithm [24, 25]. The approximate solution $U(x,t)$ at a particular time-level can be determined by using the recurrence relation, once the initial vectors c_i's have been computed from the initial and boundary conditions.

5. THE INITIAL VALUES

In order to determine the required unknowns, following associations are used

$$u_x(a,t) = 0, \; u_x(b,t) = 0,$$
$$u_{xx}(a,t) = 0, \; u_{xx}(b,t) = 0,$$

On simplifying we get a $(N+1)\times(N+1)$ matrix system

$$\begin{bmatrix} 54 & 60 & 6 & 0 & 0 & 0 & 0 & 0 \\ 101/4 & 135/2 & 105/4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 & 0 & 0 \\ 0 & 1 & 26 & 66 & 26 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & 26 & 66 & 26 & 1 \\ 0 & 0 & 0 & 0 & 1 & 105/4 & 135/2 & 101/4 \\ 0 & 0 & 0 & 0 & 0 & 6 & 60 & 54 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ . \\ \vdots \\ \vdots \\ c_{N-2} \\ c_{N-1} \\ c_N \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ . \\ \vdots \\ \vdots \\ f(x_{N-2}) \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix} \quad (18)$$

solving this matrix initial values are obtained.

6. STABILITY ANALYSIS

The stability of the scheme is investigated by using the von-Neumann method. To implement the method, linearize the nonlinear term u^3 by considering u^2 as a local constant in equation (5). Upon discretization as discuissed above lead to the system as:

$$u^{n+1}\left(1+\frac{\Delta t}{2}\left[(u^2)^n+(1-\alpha)\right]\right)+u_{xx}^{n+1}(\Delta t)+u_{xxxx}^{n+1}\left(\frac{\Delta t}{2}\right)=$$
$$u^{n}\left(1-\frac{\Delta t}{2}\left[(u^2)^n+(1-\alpha)\right]\right)-u_{xx}^{n}(\Delta t)-u_{xxxx}^{n}\left(\frac{\Delta t}{2}\right). \quad (19)$$

Now substituting the values of $u(x,t)$ and its derivatives in terms of the approximate solution, the equation can be written in terms of unknown time parameters $c_m's$, as

$$c_{m-2}^{n+1}b_1+c_{m-1}^{n+1}b_2+c_m^{n+1}b_3+c_{m+1}^{n+1}b_2+c_{m+2}^{n+1}b_1=c_{m-2}^{n}b_4+c_{m-1}^{n}b_5+c_m^{n}b_6+c_{m+1}^{n}b_5+c_{m+2}^{n}b_4. \quad (20)$$

Here,

$$b_1=1+s+y+z,\ \ b_2=26(1+s)+2y-4z,\ \ b_3=66(1+s)-6y+6z,$$
$$b_4=1-s-y-z,\ \ b_5=26(1-s)-2y+4z,\ \ b_6=66(1-s)+6y-6z,$$
$$s=\frac{\Delta t}{2}\left[(u^2)^n+(1-\alpha)\right]$$

and y, z have their predefined definition given in Section 4 as $y=\frac{20\Delta t}{h^2},\quad z=\frac{60\Delta t}{h^4},$

Now substituting,

$$c_m^n=\xi(t_n)\exp(im\beta h)$$

in equation (20), where ξ is the amplification factor, β is the mode number, h is the element size and $i=\sqrt{-1}$, we obtain

$$\xi(t_{n+1})(e^{i(m-2)\phi}b_1+e^{i(m-1)\phi}b_2+e^{im\phi}b_3+e^{i(m+1)\phi}b_2+e^{i(m+2)\phi}b_1)$$
$$=\xi(t_n)(e^{i(m-2)\phi}b_4+e^{i(m-1)\phi}b_5+e^{im\phi}b_6+e^{i(m+1)\phi}b_5+e^{i(m+2)\phi}b_4). \quad (21)$$

where $\beta h=\phi$

On dividing both sides of equation (21) by $e^{im\phi}$ we get

$$\xi(t_{n+1})\{b_1(e^{2i\phi}+e^{-2i\phi})+b_2(e^{i\phi}+e^{-i\phi})+b_3\}$$
$$=\xi(t_n)\{b_4(e^{2i\phi}+e^{-2i\phi})+b_5(e^{i\phi}+e^{-i\phi})+b_6\}. \quad (22)$$

Equation (22) can be written as

$$\frac{\xi(t_{n+1})}{\xi(t_n)}=\frac{\left[(2b_5)Cos\phi+(2b_4)Cos2\phi+b_6\right]}{\left[(2b_2)Cos\phi+(2b_1)Cos2\phi+b_3\right]}. \quad (23)$$

The condition for the scheme to be unconditionally stable is $\left|\frac{\xi(t_{n+1})}{\xi(t_n)}\right|\le 1.$

Let, $\frac{\xi(t_{n+1})}{\xi(t_n)}=\frac{A}{B}$

Then for stability

$$\left|\frac{A}{B}\right|\le 1 \Rightarrow -1\le\frac{A}{B}\le 1 \ \text{ or} -B\le A\le B. \quad (24)$$

i.e.

$$B-A\ge 0 \text{ and } A+B\ge 0. \quad (25)$$

Substituting the values of A and B from equation (23) in the expressions given by equation (25) we get,

$$A+B=26\cos\phi+2\cos^2\phi+32. \quad (26)$$

$$B-A=(13s+y-2z)\cos\phi+(s+y+z)\cos^2\phi+16s-2y+z. \quad (27)$$

It can be seen that $(A+B)$ is positive for all values of ϕ.

To show $B-A\geq 0$, we have obtained the expression for $(B-A)$ at different values of $\cos\phi$ (ranging between the max and min value i.e. +1 and -1) and are tabulated in Table 2.

Since $z=\frac{3y}{h^2}$ and $s=\frac{\Delta t}{2}\left[(u^2)^n+(1-\alpha)\right]\geq 0$ due to u^2 and $1-\alpha\geq 0$ for $\alpha\leq 1$.

Hence from the table it is evident that for all values of $\cos\phi$, $B-A\geq 0$.

Thus the scheme satisfies the conditions given by equation (25) and is unconditionally stable.

7. NUMERICAL RESULTS

Solution to the Swift–Hohenberg equation (6) is obtained with initial condition

$$u(x,t_0)=A\sin\left(\frac{\pi x}{L}\right), \quad x\in[0,L] \quad (28)$$

and boundary conditions (7) and (8) with $g_0=g_1=0$.

To gain insight into the behavior of different parameters α and L on the solution profile a series of numerical simulations are carried out. For a fix value of α, size of domain is increased and the change in the pattern is noted.

Computed results are obtained by taking time-step $\Delta t=0.01$ and the number of partitions as 100. Solutions are depicted graphically and are compared with those already available in the literature with $A=\frac{1}{5}$, .

The results of simulations are obtained for three different values of $\alpha=0.1,0.5$ and 0.9, for each of these value the domain varies from L = 4 to 10 and the solution profiles are obtained at time T=2, 4, 6, 8, 10.The profiles show the same behavior as plotted by Peletier and Rottschäfer [5] and Talay Akyildiz [13].

Case $\alpha=0.1$: Figs. 1-2 are plotted for L=4 and L=10. It is noted that for small value of parameter L=4, the solution become is tending to the trivial solution as time increases but there is a sudden change in the pattern with change in domain. For L=10 the solution begin to grow as the time extends with two zeros.

Case $\alpha=0.5$: From Figures 3-5 it can be seen that for all different values of L the solution has different nontrivial final states. For L=4 the solution has no zeros, for L=6 solution has one zero and for L=10 solution has two zeros.

Case $\alpha=0.9$: It is noted from Figs. 6-8 that for all different values of L the solution tends to converge to the final state as the time increases but is fast as compared to $\alpha=0.5$ with the similarity in number of zeros for the same domain.

The parameter A in the initial condition also plays an important role. To show the effect of parameter A, Figs. 9-10 are plotted for $\alpha = 0.5$, at time T=6 for L=6 and L=10 respectively by taking $A = 1/100, 1/50, 1/20, 1/10, 1/5$. From the graphs it is evident that for a particular value of domain at a given time-level solution behaves in the similar way for all values of *A*. But the solution profiles tends to trivial solution with increase in the value of *A* at small domain L=6 with one zero while for domain L=10 the solution tends to nontrivial solution from the trivial solution as value of *A* increases. Thus in both the cases different behaviors are shown.

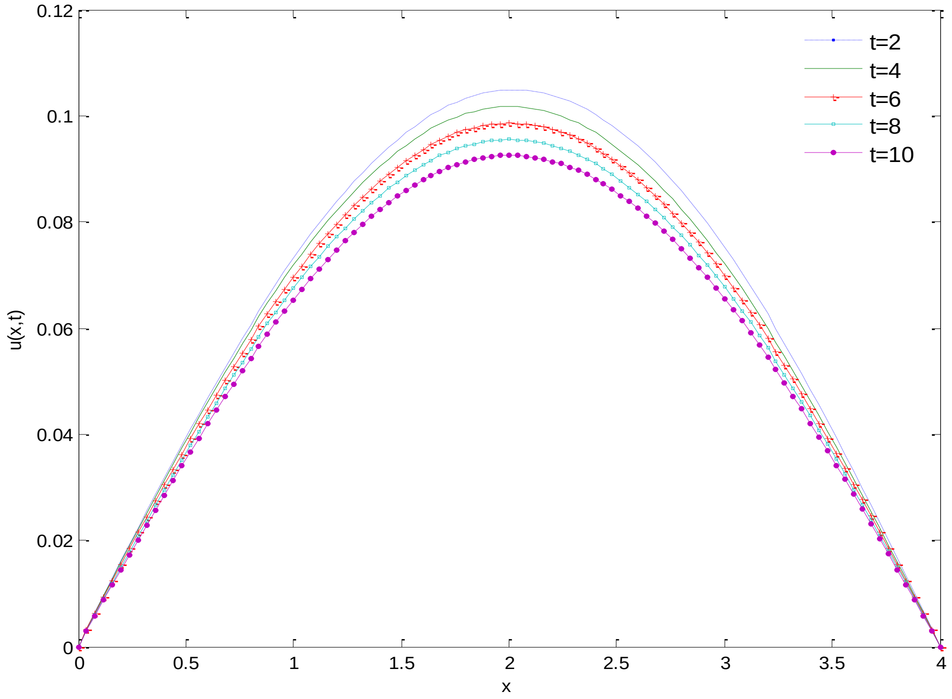

Figure 1: Time dependent profiles for L=4 for $\alpha = 0.1$ and $A = 1/5$

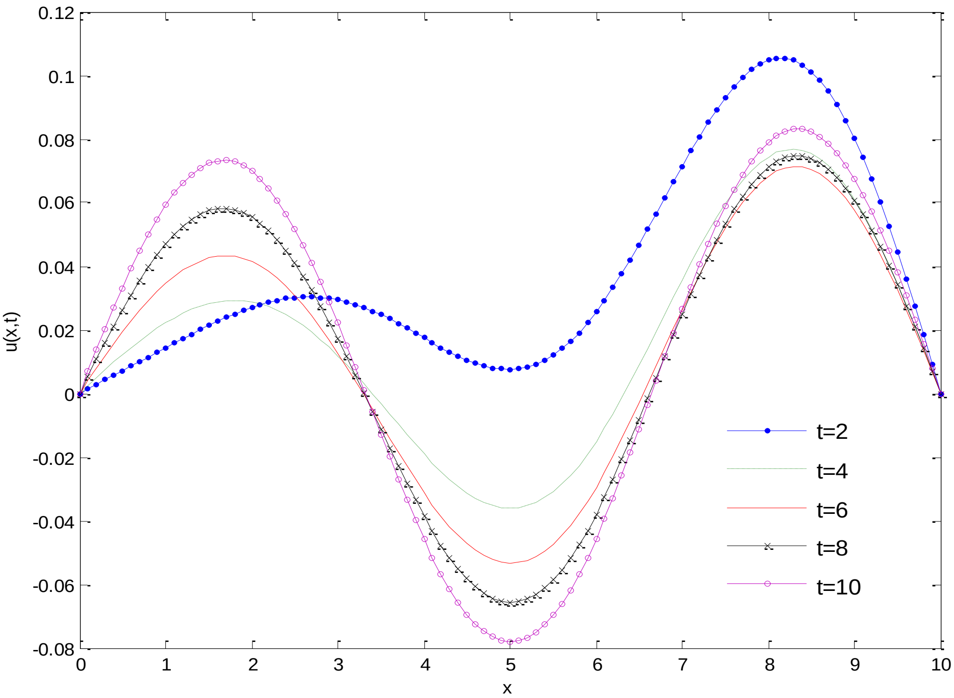

Figure 2: Time dependent profiles for L=10 for $\alpha = 0.1$ and $A = 1/5$

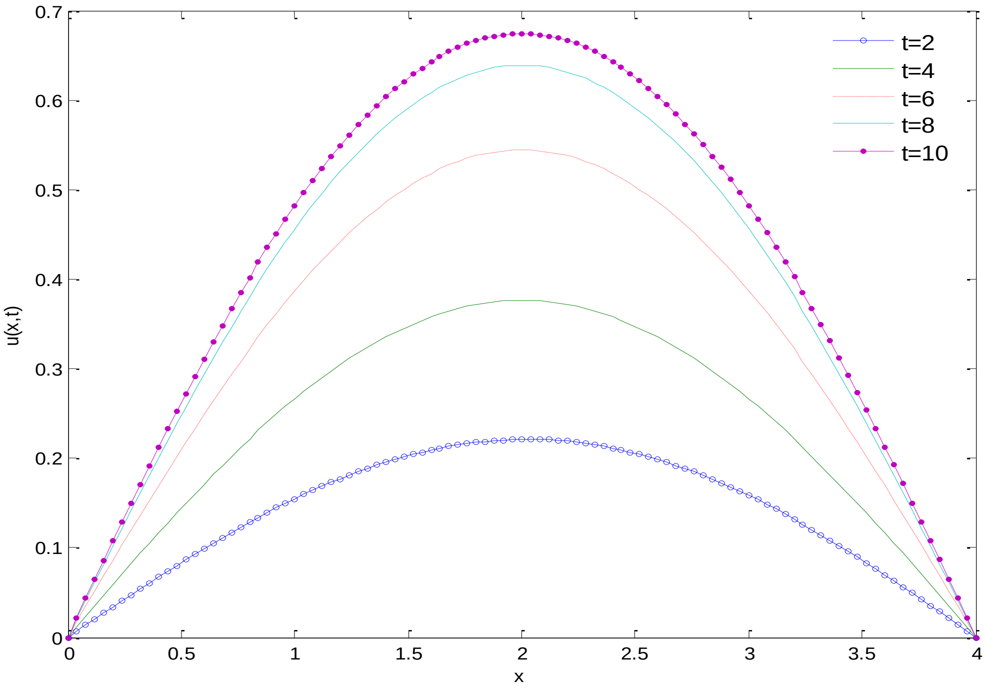

Figure 3: Time dependent profiles for L=4 for $\alpha = 0.5$ and $A = 1/5$

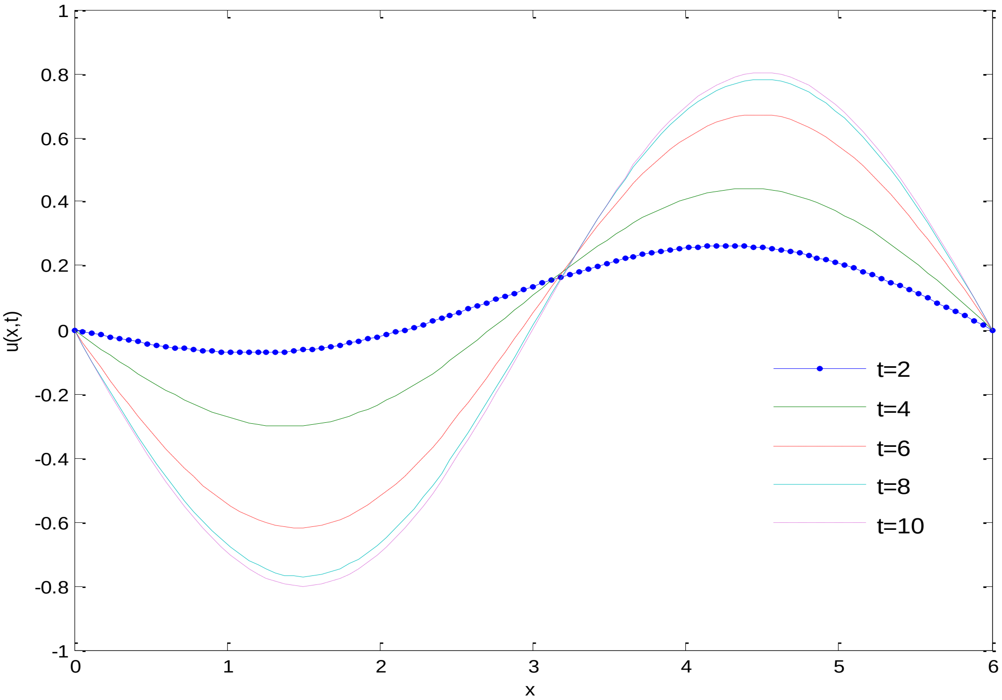

Figure 4: Time dependent profiles for L=6 for $\alpha = 0.5$ and $A = 1/5$

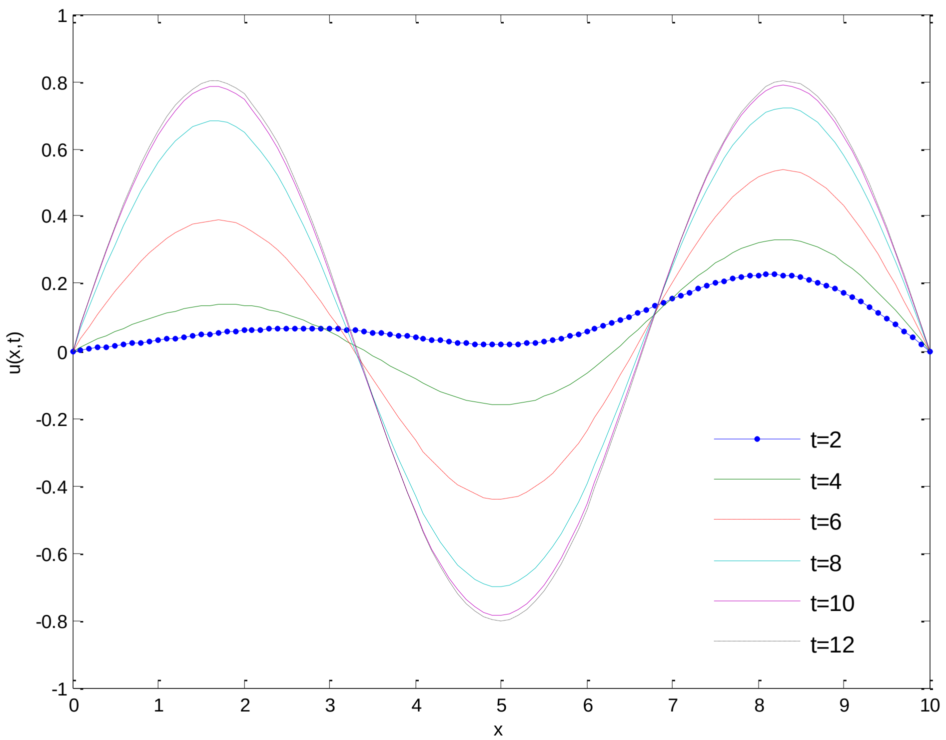

Figure 5: Time dependent profiles for L=10 for $\alpha = 0.5$ and $A = 1/5$

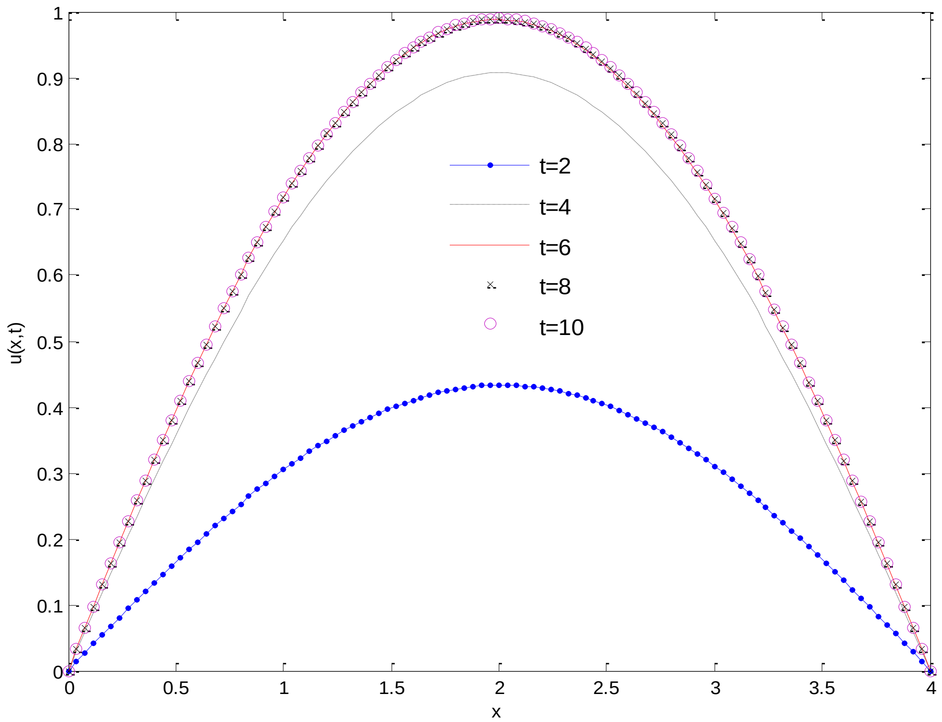

Figure 6: Time dependent profiles for L=4 for $\alpha = 0.9$ and $A = 1/5$

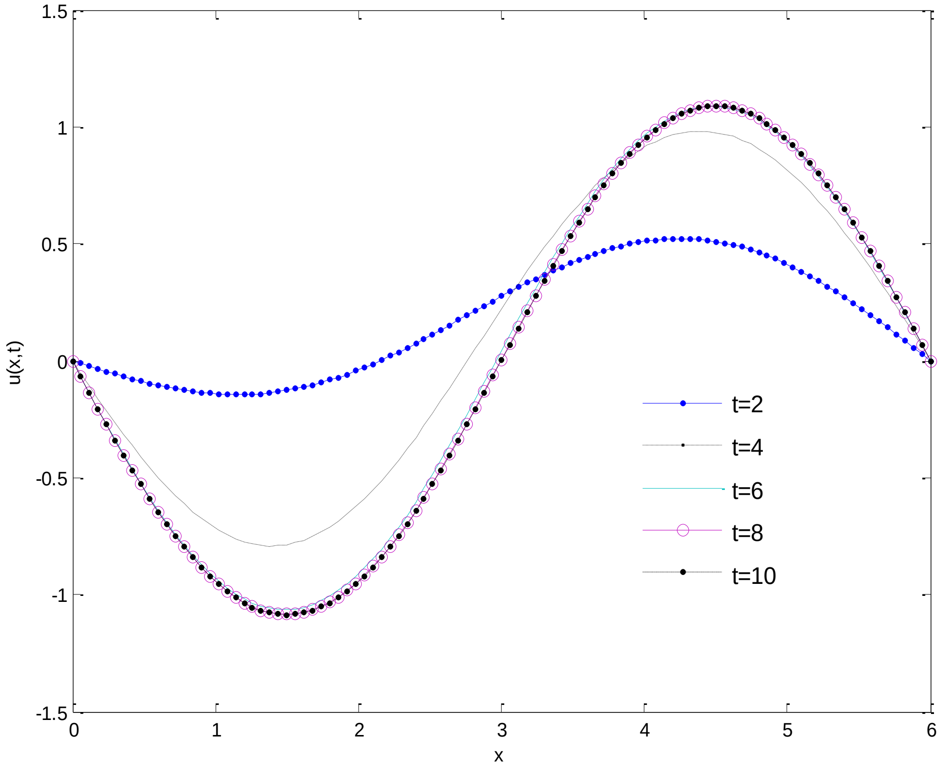

Figure 7: Time dependent profiles for L=6 for $\alpha = 0.9$ and $A = 1/5$

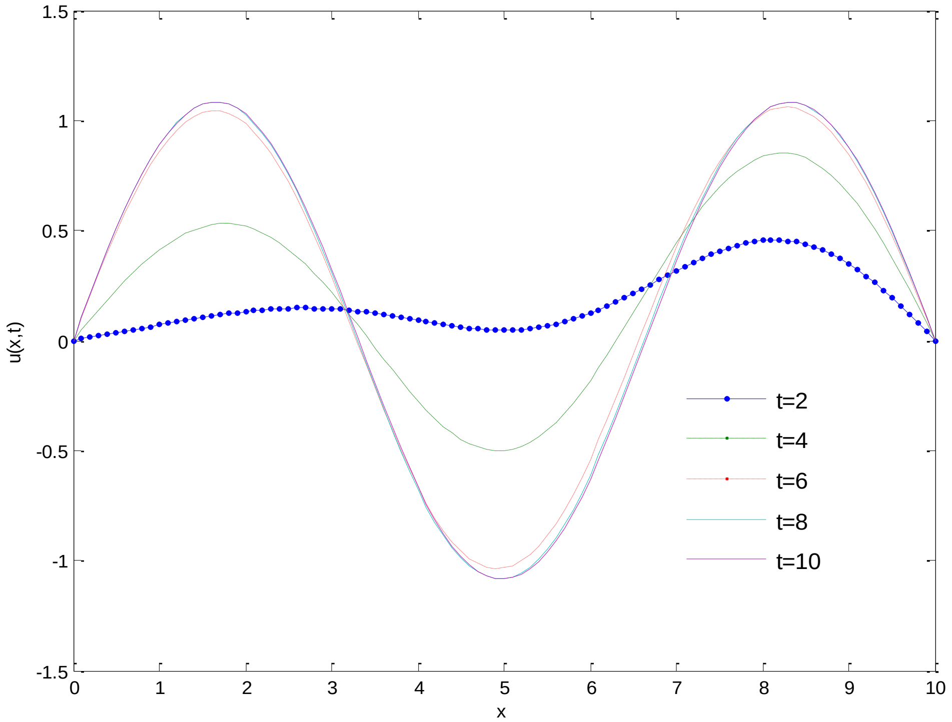

Figure 8: Time dependent profiles for L=10 for $\alpha = 0.9$ and $A = 1/5$

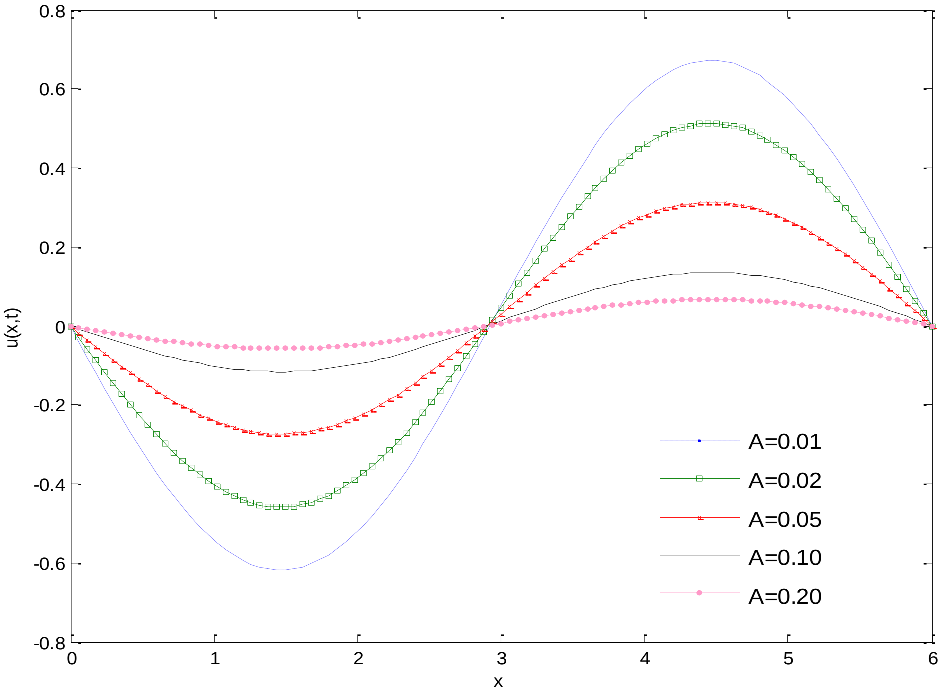

Figure 9: Time dependent profiles for L=6 for $\alpha = 0.5$ at time t=6 for different values of parameter A.

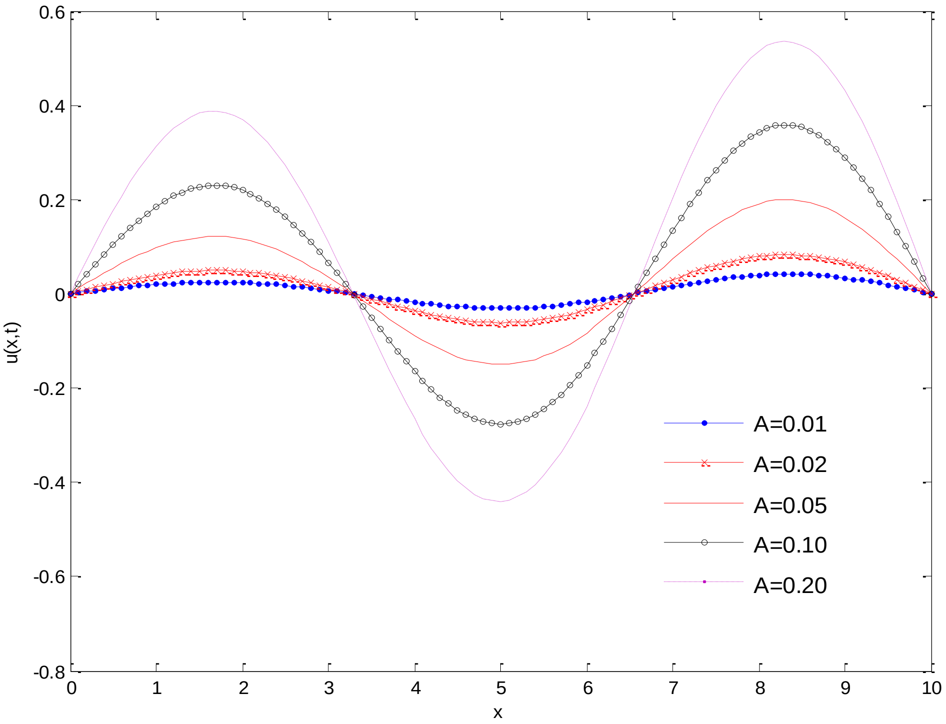

Figure 10: Time dependent profiles for L=10 for $\alpha = 0.5$ at time t=6 for different values of parameter A.

8. Concluding Remarks

In the present work the Swift-Hohenberg equation is solved numerically using collocation approach using the quintic B-splines. The nonlinear term in the equation is linearized using quasilinearization formula. To deal with the stability of the nonlinear equation using von-Neumann method, the nonlinear term in the equation is first linearized and then checked for stability. The solution obtained is presented graphically at various time intervals for different values of parameters that affect the solution of the equation.

ACKNOWLEDGEMENT: Authors are thankful to the anonymous referees for the review.

REFERENCES

[1] J.B. Swift and P.C. Hohenberg, Hydrodynamic fluctuations at the convective instability, Phys. Rev. A, 15 (1977) 319-328.

[2] P.C. Hohenberg and J.B. Swift, Effects of additive noise at the onset of Rayleigh–Bénard convection, Phys. Rev. A, 46 (1992) 4773-4785.

[3] Y. Pomeau and P. Manneville, Wave length selection in cellular flows, Phys. Lett. A, 75 (1980) 296-298.

[4] J. V. Lega, Moloney and A.C. Newell, Swift–Hohenberg equation for lasers, Phys. Rev. Lett., 73 (1994) 2978-2981.

[5] L.A. Peletier and V. Rottschäfer, Pattern selection of solutions of the Swift–Hohenberg equation, Phys. D, 194 (2004) 95-126.

[6] R.A. Gardner and C.K.R.T. Jones, Traveling waves of a perturbed diffusion equation arising in a phase field model, Indiana Univ. Math. J. 38 (1989) 1197.

[7] P.C. Carriaoa, L.F.O. Fariaa and O.H. Miyagakib, Periodic solutions for extended Fisher–Kolmogorov and Swift–Hohenberg equations by truncature techniques, Nonlinear Anal., 67 (2007) 3076-3083.

[8] J.Chaparova, L.A. Peletier and S. Tersian, Existence and nonexistence of nontrivial solutions of semilinear sixth order ordinary differential equations, Appl. Math. Lett., 17 (2004) 1207-1212.

[9] J. Chaparova, L.A. Peletier and S. Tersian , Existence and nonexistence of nontrivial solutions of semilinear fourth- and sixth-order ordinary differential equations, Adv. Diff. Eqns. 8 (2003) 1237-258.

[10] T. Ogawa and T. Okuda , Bifurcation analysis to Swift-Hohenberg equation with perturbed boundary conditions, RIMS Kˆokyˆuroku Bessatsu, B3(2007), 83-99, ©Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

[11] L.A. Peletier and V. Rottschäfer, Large time behaviour of solutions of the Swift–Hohenberg equation, C. R. Acad. Sci. Paris, Ser. I. 336 (2003) 225-230.

[12] D. Smets and J.B. Van Den Berg, Homoclinic Solutions for Swift-Hohenberg and Suspension Bridge Type Equations, J. Differential Equations, 184 (2002) 78-96.

[13] F. Talay Akyildiz, D.A. Siginer, K. Vajravelu and R. A. Van Gorder, Analytical and numerical results for the Swift–Hohenberg equation, Appl. Math. Comput. , 216 (2010) 221-226.

[14] Nikolay A. Kudryasha, Pavel N. Ryabay, Analytical and numerical solutions of the generalized dispersive Swift-Hohenberg equation, Vol 28 No. c Applid MMathematics and Computation, 2016

[15] B.S. Kumar,P. Danumjaya and A. Kumar, A fourth-order orthogonal spline collocation method to fourth-order boundary value problems, International Journal for Computational Methods in Engineering Science and Mechanics Volume 20, - Issue 5: Pages 460-470,2019

[16] W. K. Zahra, Rational spline-nonstandard finite difference scheme for the solution of time-fractional Swift–Hohenberg equation, Applied Mathematics and Computation Volume 343, 2019, Pages 372-387

[17] B. Sepehrian, M. Lashani, A numerical Solution of the Burgers Equation Using Quintic B-Splines, Proceedings of the World Congress on Engineering 2008 Vol III WCE 2008, London, U.K.
[18] I. Da˘g, B. Saka and D. Irk, Galerkin method for the numerical solution of the RLW equation using quintic B-splines, Comput. Appl. Math. 190 (2006) 532–547
[19] S.I. Zaki, A quintic B-spline finite elements scheme for the KdVB equation, Comput. Methods, Appl. Engrg. 188 (2000)121–134.
[20] G. A. Gardner, L. R. T. Gardner and A. H. A. Ali, Modelling solitons of the Korteweg–de Vries equation with quintic B-splines, U.C.N.W. Math., Preprint, 1990.
[21] R.C. Mittal and G. Arora, Quintic B-Spline Collocation Method for Numerical Solution of the Extended Fisher-Kolmogorov Equation, Int. J. of Appl. Math and Mech. 6 (1), 74-85, (2010).
[22] R.C. Mittal and G. Arora, Quintic B-spline collocation method for numerical solution of the Kuramoto–Sivashinsky equation, Commun Nonlinear Sci Numer Simulat., 15, 2798-2808, (2010).
[23] Prenter P.M., Splines and variational methods, John Wiley & Sons, New York, Inc.1975.
[24] Mukherjee, R., Huang, Z. F., & Nadgorny, B. (2014). Multiple percolation tunneling staircase in metal-semiconductor nanoparticle composites. Applied Physics Letters, *105*(17), 173104.
[25] Von Rosenberg DU. Methods for solution of partial differential equations, vol. 113. New York: American Elsevier Publishing Inc.; 1969.