Generalization of sum of positive integral powers of natural numbers

B.Mahaboob1, Y.Harnath2, C.Narayana3, V.B.V.N.Prasad4, Y. Hari Krishna5, G.Balaji Prakash6

1Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur A.P. India. E-mail: bmahaboob750@gmail.com
2Department of Mathematics, Audisankara College of Engineering & Technology (Autonomous), Gudur, SPSR Nellore (Dt), A.P India. E-mail: harnath.veddala@gmail.com
3Department of Mathematics, Sri Harsha Institute of PG Studies, SPSR Nellore(Dt), A.P India. E-mail: nareva.nlr@gmail.com
4Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur A.P. India. E-mail: vbvnprasad@kluniversity.in
5Department of Mathematics, ANURAG Engineering College, Anathagiri (v), Kodad, Suryapet, Telangana, India. E-mail: varaganihari@gmail.com
6Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur, A.P. India. E-mail: balajiprakashgudala@gmail.com

Abstract

Sum of positive integral powers of first n natural numbers has been an interesting problem for many years. Mathematicians, students and research scholars have been attempting to crack this problem for decades. The primary objective of this talk is to generate a generalized result for an ancient interesting problem in the research field of Analytic Number Theory. That problem states that sum of kth powers of first n natural number coincides with a unique a polynomial of degree (k+1) in n over natural numbers. The existence and uniqueness of this polynomial are established using the principles of Linear Algebra. The innovative result derived here opens a way to write the formula for the sum of any positive integral power of first n natural numbers.

Keywords: Rank of a Matrix, Simultaneous Nonhomogeneous Linear Equations, Cramer's Rule, Binomial Coefficients, Coefficient matrix

1. Introduction:

Thomas Harlot (1560-1621) was the first mathematician who gave the generalized form of sum of positive integral powers of first n natural numbers. Johann Faulhaber (1580-1635), a German mathematician, proposed formulas up to 17th power and his work was considered a master piece at that time. However Johann Faulhaber failed to generalize his results. Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1705) were credited with the innovation of these results in explicit form. But Jacob Bernoulli (1654-1705) gave the most significant generalized formula explicitly. In 2012, Dohyoung Ryang and Tony Thompson, in their research article, generated a formula for sum of positive integral powers of first n natural numbers. Janet Beery, in 2010, in his paper, discussed the sum of positive integral powers of first n – natural numbers. Do Tan Si, in 2019, in research article proposed tables to compute Bernoulli numbers which are used in the generalization of sum of positive integral powers of first n natural numbers.
2. Existence and uniqueness of the generalized result:

Suppose k, n are positive integers and

\[ \sum_{r=1}^{n} r^k = 1^k + 2^k + 3^k + \ldots + n^k = A_0n^{k+1} + A_1n^k + A_2n^{k-1} + \ldots + A_k n + A_{k+1} \]

Here an assumption is being made that sum of k-th powers of first n natural numbers coincides with a polynomial of degree (k+1) in n over natural numbers. Replace n by n+1

\[ \sum_{r=1}^{n+1} r^k = 1^k + 2^k + 3^k + \ldots + (n+1)^k = A_0(n+1)^{k+1} + A_1(n+1)^k + A_2(n+1)^{k-1} + \ldots + A_k n + A_{k+1} \]

Subtracting the latter from the former

\[ (n+1)^k = A_0\left[(n+1)^{k+1} - n^{k+1}\right] + A_1\left[(n+1)^k - n^k\right] + A_2\left[(n+1)^{k-1} - n^{k-1}\right] + A_3\left[(n+1)^{k-2} - n^{k-2}\right] + \ldots + A_k\left[(n+1)^1 - n^1\right] + A_{k+1}\left[(n+1)^0 - n^0\right] \]

\[ = A_0\left[(k+1)c_1 n^k + (k+1)c_1 n^{k-1} + \ldots + (k+1)c_1 n + 1\right] + A_1\left[ k_1 n^{k-1} + k_2 n^{k-2} + k_3 n^{k-3} + \ldots + k_{k-1} n + 1\right] + A_2\left[ (k-1)c_1 n^{k-2} + (k-1)c_2 n^{k-3} + \ldots + (k-1)c_{k-2} n + 1\right] + A_3\left[ (k-2)c_1 n^{k-3} + (k-2)c_2 n^{k-4} + \ldots + (k-2)c_{k-3} n + 1\right] + \ldots + A_{k-1}\left[ 2c_1 n + 1\right] + A_k \]
= \left[ A_0 \left( k + 1 \right) c_1 \right] n^k + \left[ A_0 \left( k + 1 \right) c_2 \right] + A_1 k c_1 \right] n^{k-1} \\
+ \left[ A_0 \left( k + 1 \right) c_3 \right] + A_1 k c_2 + A_2 \left( k - 1 \right) c_1 \right] n^{k-2} \\
+ \left[ A_0 \left( k + 1 \right) c_4 \right] + A_1 k c_3 + A_2 \left( k - 1 \right) c_2 + A_3 \left( k - 2 \right) c_1 \right] n^{k-3} \\
+ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
+ \left[ A_0 \left( k + 1 \right) c_k \right] + A_1 k c_{k-1} + A_2 \left( k - 1 \right) c_{k-2} + A_3 \left( k - 2 \right) c_{k-3} + \ldots \ldots + A_{k-1} \cdot 2 c_1 \right] n \\
+ \left[ A_0 \left( k + 1 \right) c_{k+1} \right] + A_1 k c_k + A_2 \left( k - 1 \right) c_{k-1} + A_3 \left( k - 2 \right) c_{k-2} + \ldots \ldots + A_{k-1} \cdot 2 c_{k-1} + A_k 1 c_1 \right]

Applying binomial expansion for LHS and comparing the like powers of $n^k$ one can get

\[ k_{c_0} = A_0 \left( k + 1 \right) c_1 \] ........................ (1)

\[ k_{c_1} = A_0 \left( k + 1 \right) c_2 + A_1 k c_1 \] ........................ (2)

\[ k_{c_2} = A_0 \left( k + 1 \right) c_3 + A_1 k c_2 + A_2 \left( k - 1 \right) c_1 \] ........................ (3)

\[ k_{c_3} = A_0 \left( k + 1 \right) c_4 + A_1 k c_3 + A_2 \left( k - 1 \right) c_2 + A_3 \left( k - 2 \right) c_1 \] ........................ (4)

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ k_{c_i} = A_0 \left( k + 1 \right) c_{i+1} + A_1 k c_{i+1} + A_2 \left( k - 1 \right) c_{i+2} + \ldots \ldots + A_{i-1} 1 c_1 \] \[ k_{c_{i+1}} = A_0 \left( k + 1 \right) c_i + A_1 k c_{i+1} + A_2 \left( k - 1 \right) c_i + \ldots \ldots + A_{i-1} 2 c_i \]

\[ k_{c_{i+2}} = A_0 \left( k + 1 \right) c_{i+2} + A_1 k c_{i+2} + A_2 \left( k - 1 \right) c_{i+3} + \ldots \ldots + A_{i-1} 2 c_{i+1} + A_i 1 c_i \] ........................ (k+1)

The (k+1) equations constitute a system of linear non-homogenous equations in (k+1)-unknowns $A_0, \ldots , A_k$. In matrix algebra notations this system is written as

\[
\begin{bmatrix}
(k+1)_{c_1} & 0 & 0 & \ldots & 0 & 0 \\
(k+1)_{c_2} & k_{c_3} & 0 & \ldots & 0 & 0 \\
(k+1)_{c_3} & k_{c_2} & k-1_{c_1} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(k+1)_{c_i} & k_{c_{i+1}} & k-1_{c_{i+2}} & \ldots & 2 c_i & 0 \\
(k+1)_{c_{i+1}} & k_{c_i} & k-1_{c_{i+2}} & \ldots & 2 c_2 & 1 c_1 \\
\end{bmatrix}
= 
\begin{bmatrix}
A_0 \\
A_1 \\
A_2 \\
\vdots \\
A_{i-1} \\
A_i \\
\end{bmatrix}

= 
\begin{bmatrix}
k_{c_0} \\
k_{c_1} \\
k_{c_2} \\
\vdots \\
k_{c_{i+1}} \\
k_{c_i} \\
\end{bmatrix}


\[ \begin{bmatrix}
(k+1)_{c_1} & 0 & 0 & . & 0 & 0 \\
(k+1)_{c_2} & k_{c_1} & 0 & . & 0 & 0 \\
(k+1)_{c_3} & k_{c_2} & k-1_{c_1} & . & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
(k+1)_{c_{k+1}} & k_{c_{k-1}} & k-1_{c_{k-2}} & 2_c_1 & 0 \\
\end{bmatrix}
\]

\[ \Delta = \text{Determinant of Coefficient Matrix} \]

\[ \Delta_0, \Delta_1, \ldots, \Delta_k \] are obtained by replacing 1\text{st}, 2\text{nd} \ldots (k+1)\text{th} columns by the column matrix

\[ \begin{bmatrix}
k_{c_0} \\
k_{c_1} \\
k_{c_2} \\
\vdots \\
k_{c_{k+1}} 
\end{bmatrix} \]

\[ \Delta = k + 1_{c_1} k_{c_1} k-1_{c_1} \ldots 2_{c_1} 1_{c_1} \]

\[ = (k+1)! \neq 0 \text{ for any } k \in N \]

Since determinant of coefficient matrix is non zero and the number of equations is equal to number of unknowns, the above system of (k+1) –equations have a unique solution.

3. Formulas to compute the coefficients by Crammer’s rule:

\[ A_0 = \frac{\Delta_0}{\Delta} \]
\[ A_1 = \frac{\Delta_1}{\Delta} \]
\[ A_2 = \frac{\Delta_2}{\Delta} \]
\[ \vdots \]
\[ A_k = \frac{\Delta_k}{\Delta} \]

By convention \( A_{k+1} \) is taken as 0.
As $\Delta$ is non-zero the above formulas prove the existence of the coefficients $A_0, A_1, A_2, \ldots, A_k$.

4. Special Cases

**Case i) (For $k=1$)**

$$\sum_{r=1}^{n} r^1 = 1^1 + 2^1 + 3^1 + \ldots + n^1 = \sum n$$

$$1_{c_0} = A_0, 2_{c_1} \Rightarrow 1 = A_0, 2 \Rightarrow A_0 = \frac{1}{2}$$

$$1_{c_1} = \frac{1}{2}, 2_{c_2} + A_1, 1_{c_1} \Rightarrow A_1 = \frac{1}{2}$$

$$\therefore 1^1 + 2^1 + 3^1 + \ldots + n^1 = \frac{1}{2} n^2 + \frac{1}{2} n = \frac{n(n+1)}{2}$$

**Case ii) (For $k=2$)**

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum n^2 = A_0 n^3 + A_1 n^2 + A_2 n$$

$$2_{c_0} = A_0, 3_{c_2} \Rightarrow A_0 = \frac{1}{3}$$

$$2_{c_1} = A_0, 3_{c_3} + A_1, 2_{c_2} \Rightarrow 2 = A_0, 3 + A_1, 2 \Rightarrow A_1 = \frac{1}{2}$$

$$2_{c_2} = A_0, 3_{c_4} + A_1, 2_{c_3} + A_2, 1_{c_1}$$

$$1 = A_0 + A_1 + A_2 \Rightarrow A_2 = \frac{1}{6}$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n = \frac{n(n+1)(2n+1)}{6}$$

**Case iii) (For $k=3$)**

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = A_0 n^4 + A_1 n^3 + A_2 n^2 + A_3 n$$

$$3_{c_0} = A_0, 4_{c_4} \Rightarrow A_0 = \frac{1}{4}$$
\[ 3c_1 = A_0 4c_2 + A_1 3c_1 \Rightarrow A_1 = \frac{1}{2} \]

\[ 3c_2 = A_0 4c_3 + A_1 3c_2 + A_2 2c_1 \Rightarrow A_2 = \frac{1}{4} \]

\[ 3c_3 = A_0 4c_4 + A_1 3c_3 + A_2 2c_2 + A_3 1c_1 \Rightarrow A_3 = 0 \]

\[
\therefore 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2 = \frac{n^2(n+1)^2}{4}
\]

**Case iv) (For k=4)**

\[
\sum_{r=1}^{n} r^4 = 1^4 + 2^4 + 3^4 + \ldots + n^4 = A_0 n^5 + A_1 n^4 + A_2 n^3 + A_3 n^2 + A_4 n
\]

\[ 4c_0 = A_0 5c_1 \Rightarrow A_0 = \frac{1}{5} \]

\[ 4c_1 = A_0 5c_2 + A_1 4c_1 \Rightarrow 4 = A_0 10 + A_1 4 \]

\[ \Rightarrow 4 = 2 + 4A_1 \Rightarrow A_1 = \frac{1}{2} \]

\[ 4c_2 = A_0 5c_3 + A_1 4c_2 + A_2 3c_1 \]

\[ 6 = 2 + 3 + A_2 (3) \Rightarrow A_2 = \frac{1}{3} \]

\[ 4c_3 = A_0 5c_4 + A_1 4c_3 + A_2 3c_2 + A_3 2c_1 \]

\[ 4 = 1 + 2 + 1 + A_3 (2) \Rightarrow A_3 = 0 \]

\[ 4c_4 = A_0 + A_1 + A_2 + A_3 + A_4 \]

\[ 1 = \frac{1}{5} + \frac{1}{2} + \frac{1}{3} + A_4 \Rightarrow A_4 = 1 - \left( \frac{1}{5} + \frac{1}{2} + \frac{1}{3} \right) \]

\[ A_4 = -\frac{1}{30} \]

\[ 1^4 + 2^4 + 3^4 + \ldots n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{1}{30} n \]

**Case v) (For k=5)**

\[
\sum_{r=1}^{n} r^5 = 1^5 + 2^5 + 3^5 + \ldots + n^5 = A_0 n^6 + A_1 n^5 + A_2 n^4 + A_3 n^3 + A_4 n^2 + A_5 n
\]

\[ 5c_0 = A_0 (6c_1) \Rightarrow A_0 = \frac{1}{6} \]
$$5_{c_1} = A_06_{c_2} + A_15_{c_1} \Rightarrow 5 = A_1 = \frac{1}{2}$$

$$5_{c_2} = A_06_{c_3} + A_15_{c_2} + A_24_{c_1} \Rightarrow A_2 = \frac{5}{12}$$

$$5_{c_3} = A_06_{c_4} + A_15_{c_3} + A_24_{c_2} + A_33_{c_1}$$

$$10 = \frac{5}{2} + \frac{5}{2} + 3A_3 \Rightarrow A_3 = 0$$

$$5_{c_4} = A_06_{c_5} + A_15_{c_4} + A_24_{c_3} + A_33_{c_2} + A_42_{c_1}$$

$$5 = 1 + \frac{5}{2} + \frac{5}{3} + A_42 \Rightarrow 4 - \frac{5}{2} - \frac{5}{3} = 2A_4 \Rightarrow A_4 = -\frac{1}{12}$$

$$5_{c_5} = A_06_{c_6} + A_15_{c_5} + A_24_{c_4} + A_33_{c_3} + A_42_{c_2} + A_51_{c_1}$$

$$1 = \frac{1}{6} + \frac{1}{2} + \frac{5}{12} - \frac{1}{12} + A_3 \Rightarrow A_3 = 0$$

$$\therefore 1^5 + 2^5 + 3^5 + \ldots + n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

5. Simultaneous Non homogenous Liner equations:

\[
\begin{array}{cccc}
  k + 1_{c_1} & & & \\
  k + 1_{c_2} & k_{c_2} & & \\
  k + 1_{c_3} & k_{c_3} & k - 1_{c_1} & \\
  k + 1_{c_4} & k_{c_4} & k - 1_{c_2} & k - 2_{c_1} \\
  k + 1_{c_5} & k_{c_5} & k - 1_{c_3} & k - 2_{c_2} & k - 3_{c_1}
\end{array}
\]

(Table 1) (Binomial Triangle) (Coefficients)
Table 2

From Tables (1) and (2) one can write simultaneous linear non-homogenous equations.

\[ A_0(k + 1)_{c_0} = k_{c_0} \]
\[ A_0(k + 1)_{c_1} + A_1 k_{c_1} = k_{c_1} \]
\[ A_0(k + 1)_{c_2} + A_1 k_{c_2} + A_2(k - 1)_{c_2} = k_{c_2} \]
\[ \ldots \ldots \ldots \]
\[ A_0(k + 1)_{c_{k-1}} + A_1 k_{c_{k-1}} + A_2(k - 1)_{c_{k-1}} + \ldots + A_{k-1}2_{c_{k-1}} + A_k1_{c_{k-1}} = k_{c_{k-1}} \]

It is interesting to note that

\[ A_0 = \frac{1}{k + 1} \]
\[ A_1 = \frac{1}{2} \]
\[ A_2 = \frac{k}{12} \]
\[ A_3 = 0 \]
\[ A_4 = \frac{-k(k - 1)(k - 2)}{720} \] etc.

From \((k+1)^{th}\) equation, one can observe that sum of \(A_0, A_1, A_2, \ldots, A_k\) is 1.

\[ A_0 + A_1 + A_2 + \ldots + A_k = 1 \Rightarrow A_{k+1} = 0. \]

6. Conclusion and Future research:

The above conversation provides answers to the most interesting questions in the research field of Analytical Number Theory.

Q: Is the sum of positive integral powers of first \(n\) – natural numbers is a polynomial in \(n\) over natural numbers?

A: Yes

Q: Is such polynomial unique?
A: Yes.

The generalized result generated in this conversation certainly opens a way for young researchers and they can have a glance on these innovative ideas and may derive some more interesting results.

References:


