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OSCILLATION CRITERIA OF THIRD ORDER NON-LINEAR DAMPED DELAY DIFFERENCE EQUATIONS

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ABSTRACT: In this paper, we consider the third order non-linear difference equations of the form $\Delta\left(c_n\Delta\left(d_n(\Delta x_n)^{\delta}\right)\right)+p_n(\Delta x_n)^{\delta}+q_n g(x_{n-\lambda})=0, n\geq n_0>0.$ **We establish new oscillation results for the third order equation by using Riccati transformation technique. Examples are given to illustrate the importance of the results.**

KEYWORDS: Riccati transformation, Delay Difference Equation, Oscillation, Third order.
AMS Subject Classification: 39A10

Introduction

We consider non-linear third order difference equations of the form

$$\Delta\left(c_n\Delta\left(d_n(\Delta x_n)^{\delta}\right)\right)+p_n(\Delta x_n)^{\delta}+q_n g(x_{n-\lambda})=0, \qquad n\geq n_0>0 \tag{1.1}$$

where $\delta\geq 1$ is the ratio of positive odd integers.

(i) $\{c_n\},\{d_n\},\{p_n\}$ and $\{q_n\}$ are positive real sequences.

(ii) $\{g_n\}$ is a real sequence such that $ug(u)>0$ for $u\neq 0$ and $g(u)/u^{\beta}\geq k>0$ where β is ratio of positive integers.

(iii) λ is a positive integer.

By a solution of equation (1.1) we mean a real sequence $\{x_n\}$ defined for $n\geq n_0-\lambda$ and satisfies the equation (1.1) for all $n\geq n_0$. A solution $\{x_n\}$ of equation (1.1) is said to be oscillatory if is neither eventually positive nor eventually negative, and otherwise non-oscillatory. A solution $\{x_n\}$ of equation (1.1) is called non-oscillatory if all its solutions are non-oscillatory.

There are many papers dealing with oscillatory and asymptotic behavior of solutions of several classes of third order functional difference equations, see [3]-[8] , [10]- [13] and the reference cited therein. In[14] the authors considered the following the second order difference equation

$$\Delta(c_n\Delta z_n)+\frac{p_n}{q_{n+1}}z_{n+1}=0 \tag{1.2}$$

is non- oscillatory .

Very recently, in [6] the authors discussed the oscillatory and asymptotic behavior of solution of the equation

$$\Delta\left(a_n \Delta\left(b_n \left(\Delta y_n\right)^{\alpha}\right)\right) + p_n \left(\Delta y_{n+1}\right)^{\alpha} + q_n f\left(y_{n-l}\right) = 0, n \ge n_0 \qquad (1.3)$$

Equation (1.3), the authors assumed the coefficient sequence of the damping term is positive.

In section 2, we will present some lemmas which are useful in establish our main results. In section 3, we will state and prove the main results and give examples illustrate them.

2. Auxiliary Results

We define,

$L_0 x_n = x_n,\qquad L_1 x_n = d_n \left(\Delta\left(L_0 x_n\right)\right)^{\delta},\quad L_2 x_n = c_n \Delta\left(L_1 x_n\right),\qquad L_3 x_n = \Delta\left(L_2 x_n\right)$ on I.

Hence (1.1) can be written as

$$L_3 x_n + \left(\frac{p_n}{d_n}\right) L_1 x_n + q_n g\left(x_{n-\lambda}\right) = 0.$$

Remark 2.1

We denote the following notations:

$$D_1(n) = \sum_{s=n_1}^{n-1} \frac{1}{\left(d_s\right)^{1/\delta}},\qquad D_2(n) = \sum_{s=n_1}^{n-1} \frac{1}{\left(c_S\right)}, \text{ and } \qquad D^*(n) = \sum_{s=n_1}^{n-1} \left[\frac{D_2(s)}{d_s}\right]^{1/\delta}$$

for $n_0 \le n_1 \le n < \infty$,

We assume that $\lim_{n\to\infty} D_1(n) = \infty$ as $n \to \infty$ (2.1)

and $\lim_{n\to\infty} D_2(n) = \infty$ as $n \to \infty$ (2.2)

Lemma 2.2 Suppose that (1.2) is non-oscillatory. If $\{x_n\}$ is a non-oscillatory solution of (1.1) on $[n_1, \infty), n_1 \ge n_0$, then there exists $n_2 \in [n_1, \infty)$ such that $x_n L_1 x_n > 0$ or $x_n L_1 x_n < 0$ for $n \ge n_2$.

Proof: The proof is similar to that of Lemma 2.1 in [14] and hence the details are omitted.

Lemma 2.3 Let $\{x_n\}$ be a non-oscillatory solution of (1.1) with $x_n L_1 x_n > 0$ for $n \ge n_1 \ge n_0$ then

$L_1 x_n \ge D_2(n) L_2 x_n$ for all $n \ge n_1$ (2.3) and

$x_n \geq D^*(n) L_2^{1/\delta} x_n$ for all $n \geq n_1$ (2.4)

Proof: Let $\{x_n\}$ be a non-oscillatory solution of (1.1) say $x_n > 0,\ x_{n-\lambda} > 0$ and $L_1 x_n > 0$ for $n \geq n_1 \geq n_0$. Since $L_3 x_n = -\left(\frac{p_n}{d_n}\right) L_1 x_n - q_n g(x_{n-\lambda}) \leq 0.$

we have that $L_2 x_n$ is non increasing on $[n_1, \infty)$, and hence

$$L_1 x_n = L_1 x_{n_1} + \sum_{s=n_1}^{n-1} \Delta(L_1 x_s) \geq \sum_{s=n_1}^{n-1} \Delta(L_1 x_s)$$

$$= \sum_{s=n_1}^{n-1} \frac{L_2 x_s}{c_s} \geq \left[\sum_{s=n_1}^{n-1} \frac{1}{c_s}\right] L_2 x_n$$

$$= L_2 x_n D_2(n)$$

This implies

$$\Delta x_n \geq \left[\frac{D_2(n)}{d_n}\right]^{1/\delta} (L_2 x_n)^{1/\delta}.$$

Now, summing this inequality from n_1 to $n-1$ and using the fact that $L_2 x_n$ is non increasing, we find

$$x_n = x_{n_1} + \sum_{s=n_1}^{n-1} \Delta x_s \geq \sum_{s=n_1}^{n-1} \Delta x_s$$

$$\geq \sum_{s=n_1}^{n-1} \left[\frac{D_2(s)}{d_s}\right]^{1/\delta} (L_2 x_s)^{1/\delta}$$

$$\geq \left[\sum_{s=n_1}^{n-1} \left[\frac{D_2(s)}{d_s}\right]^{1/\delta}\right] (L_2 x_n)^{1/\delta}$$

$$= D^*(n) (L_2 x_n)^{1/\delta} \quad for\ n \geq n_1.$$

This completes the proof.

Next, the following two lemmas are consider by the second order delay difference equation

$$\Delta(c_n \Delta x_n) = Q_n x_{n-l}$$

(2.5)

where $\{Q_n\}$ is a positive real sequence and l is a positive integer.

Lemma 2.4 If

$$\limsup_{n\to\infty} \sum_{s=n-l}^{n-1} Q_s D_2(s-l) > 1$$

(2.6)

then all bounded solutions of (2.5) are oscillatory.

Proof: Let $\{x_n\}$ be a bounded non-oscillatory solution of (2.5), say $x_n > 0$ and $x_{n-l} > 0$ for

$n \geq n_1$

for some $n_1 \geq n_0$. By (2.5), $c_n \Delta x_n$ is strictly non-decreasing on $[n_1, \infty)$. Hence for any $n_2 \geq n_1$, we have

$$x_n = x_{n_2} + \sum_{s=n_2}^{n-1} \Delta x_s = x_{n_2} + \sum_{s=n_2}^{n-1} \frac{c_s \Delta x_s}{c_s}$$

$$> x_{n_2} + c_{n_2} \Delta x_{n_2} \sum_{s=n_2}^{n-1} \frac{1}{c_s}$$

$$= x_{n_2} + c_{n_2} \Delta x_{n_2} D_2(n)$$

So $\Delta x_{n_2} < 0$ as otherwise (2.2) this imply $x_n \to \infty$ as $n \to \infty$, we get a contradiction to the boundedness of x_n. Also, we get

$x_n > 0, \Delta x_n < 0$ and $\Delta(c_n \Delta x_n) > 0$ on $[n_1, \infty)$ (2.7)

Now for $v \geq u \geq n_1$, we have

$$x_u > x_u - x_v = -\sum_{s=u}^{v-1} \Delta x_s = -\sum_{s=u}^{v-1} \frac{c_s \Delta x_s}{c_s}$$

$$\geq -\left[\sum_{s=u}^{v-1} \frac{1}{c_s}\right] c_v \Delta x_v = -D_2(v) c_v \Delta x_v \qquad (2.8)$$

For $n \geq s \geq n_1$, setting $u = s - l$ and $v = n - l$ in (2.8), we get

$$x_{s-l} > -D_2(n-l) c_{n-l} \Delta x_{n-l} \qquad (2.9)$$

Summing (2.5) from $n-l$ to $n-1$ we obtain

$$-c_{n-l} \Delta x_{n-l} > c_n \Delta x_n - c_{n-l} \Delta x_{n-l}$$

$$= \sum_{s=n-l}^{n-1} Q_s x_{s-l}$$

$$\overset{(2.9)}{>} -\left[\sum_{s=n-l}^{n-1} Q_s D_2(s-l) c_{n-l} \Delta x_{n-l}\right]$$

$$\text{(i.e)}\ 1 > \sum_{s=n-l}^{n-1} Q_s D_2(s-l) \qquad (2.10)$$

Taking lim sup as $n \to \infty$ on both sides of (2.10) yields a contradiction to (2.6) and completes the proof.

Lemma 2.5 If

$$\limsup_{n \to \infty} \sum_{s=n-l}^{n-1} \left(\frac{1}{c_n} \sum_{s=u}^{n-1} Q_s \right) > 1 \qquad (2.11)$$

then all bounded solutions of (2.5) are oscillatory.

Proof: Let $\{x_n\}$ be a bounded non-oscillatory solution of equation (2.5), say $x_n > 0$ and

$x_{n-l} > 0$ for $n \geq n_1$ for some $n_1 \geq n_0$. As in lemma 2.4, we obtain (2.7) summing (2.5) from u to $n-1$, we get

$$-c_u \Delta x_u > c_n \Delta x_n - c_u \Delta x_u = \sum_{s=u}^{n-1} Q_s x_{s-l}$$

$$\geq \left[\sum_{s=u}^{n-1} Q_s \right] x_{n-l}$$

$$(i.e) - \Delta x_u > \left[\frac{1}{c_u} \sum_{s=u}^{n-1} Q_s \right] x_{n-l} \tag{2.12}$$

Summing (2.12) from $n-l$ to $n-1$, we get

$$x_{n-l} > x_{n-l} - x_n$$

$$> \left[\sum_{s=n-l}^{n-1} \left(\frac{1}{c_u} \sum_{s=u}^{n-1} Q_s \right) \right] x_{n-l}$$

$$1 > \sum_{s=n-l}^{n-1} \left(\frac{1}{c_u} \sum_{s=u}^{n-1} Q_s \right) \tag{2.13}$$

Taking lim sup as $n \to \infty$ on both sides of (2.13) yields a contradiction to (2.11) and completes the proof.

3. Oscillation Results by Riccati method

Now, we establish the main result of this paper.

Theorem 3.1 Assume that (2.1), (2.2) and $\delta \geq \beta$. Suppose (1.2) is non-oscillatory. If there exists a positive sequence $\{\rho_n\}$ such that $\rho_n > 0$ and $n - \lambda \leq n - l \leq n$ for all $n \geq n_0$ satisfying

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n-1} \left[k\rho_s q_s - \frac{A^2{}_s}{4B_s} \right] = \infty \tag{3.1}$$

for any $n_1 \in I$, where for $n \geq n_1$.

$$\begin{cases} A_n = \dfrac{\Delta\rho_n}{\rho_{n+1}} - \dfrac{\rho_n}{\rho_{n+1}} \dfrac{p_n}{d_n} D_2(n) \\ B_n = c^* \dfrac{\rho_n}{\rho_{n+1}^2} \left[\dfrac{D_2(n-\lambda)}{d_{n-\lambda}} \right]^{1/\delta} \left[D^*(n-\lambda) \right]^{\beta-1} \end{cases} \tag{3.2}$$

and (2.6) or (2.11) holds with

$$Q_n = \left[ckq_n \left(D_1(n-l) \right)^\beta - \frac{p_n}{d_n} \right] \geq 0$$

for all $n \geq n_1$ with $c, c^* > 0$ then every solution $\{x_n\}$ of (1.1) or $L_2 x_n$ is oscillatory.

Proof:

Let $\{x_n\}$ be a non-oscillatory solution of (1.1) on $[n_1,\infty), n_1 \ge n_0$. Without loss of generality, we may assume that $x_n > 0$ and $x_{n-\lambda} > 0$ for $n \ge n_1$. From lemma 2.2, it follows that $L_1 x_n < 0$ or
$L_1 x_n > 0$ for $n \ge n_1$. First, we assume $L_1 x_n > 0$ on $[n_1,\infty)$. By (1.1), $L_2 x_n$ is strictly decreasing. Hence for any $n_2 \ge n_1$, we have

$$L_1 x_n = L_1 x_{n_2} + \sum_{s=n_2}^{n-1} \Delta(L_1 x_s) = L_1 x_{n_2} + \sum_{s=n_2}^{n-1} \frac{L_2 x_s}{c_s}$$

$$\le L_1 x_{n_2} + \left[\sum_{s=n_2}^{n-1} \frac{1}{c_s}\right] L_2 x_{n_2} = L_1 x_{n_2} + L_2 x_{n_2} D_2(n) \tag{3.3}$$

So $L_2 x_{n_2} > 0$ as otherwise (2.2) would imply $L_1 x_n \to -\infty$ as $n \to \infty$, a contradiction to the positively of $L_1 x_n$. Altogether $L_2 x_n > 0$ on $[n_1,\infty)$.
Define

$$w_n = \frac{\rho_n L_2 x_n}{x_{n-\lambda}^{\beta}} \tag{3.4}$$

By using (1.1), (2.3) and the condition (ii) on *g*, we obtain

$$\Delta w_n = \frac{\rho_n}{x_{n-\lambda}^{\beta}} \Delta L_2 x_n + L_2 x_{n+1} \Delta\left(\frac{\rho_n}{x_{n-\lambda}^{\beta}}\right)$$

$$= \Delta\rho_n \frac{w_{n+1}}{\rho_{n+1}} + \frac{\Delta L_2 x_n w_n}{L_2 x_n} - \beta\rho_n \frac{\Delta x_{n-\lambda}}{x_{n-\lambda}} \frac{w_{n+1}}{\rho_{n+1}}$$

$$\overset{(1.1)}{=} \Delta\rho_n \frac{w_{n+1}}{\rho_{n+1}} - \frac{w_n\left[\frac{p_n}{d_n} L_1 x_n + q_n g(x_{n-\lambda})\right]}{L_2 x_n} - \beta\rho_n \frac{\Delta x_{n-\lambda}}{x_{n-\lambda}} \frac{w_{n+1}}{\rho_{n+1}}$$

$$\overset{(2.3)}{\le} \frac{\Delta\rho_n w_{n+1}}{\rho_{n+1}} - w_n \frac{p_n}{d_n} D_2(n) - k\rho_n q_n - \beta\rho_n \frac{\Delta x_{n-\lambda}}{x_{n-\lambda}} \frac{w_{n+1}}{\rho_{n+1}}$$

Since $L_2 x_n$ is decreasing, we have $\frac{\Delta x_{n+1-\lambda}}{\Delta x_{n-\lambda}} \ge \left(\frac{d_{n-\lambda}}{d_{n+1-\lambda}}\right)^{1/\delta}$ and $\frac{w_{n+1}}{\rho_{n+1}} \le \frac{w_n}{\rho_n}$

$$\Delta w_n \le -k\rho_n q_n - \frac{w_{n+1}}{\rho_{n+1}} \rho_n \frac{p_n}{d_n} D_2(n) + \Delta\rho_n \frac{w_{n+1}}{\rho_{n+1}} - \beta\rho_n \frac{\Delta x_{n-\lambda}}{x_{n-\lambda}} \frac{w_{n+1}}{\rho_{n+1}}$$

$$= -k\rho_n q_n + A_n w_{n+1} - \beta\rho_n \frac{\Delta x_{n-\lambda}}{x_{n-\lambda}} \frac{w_{n+1}}{\rho_{n+1}} \tag{3.5}$$

From the definition of $L_1 x_n$ and (2.3), we get

$$\Delta x_{n-\lambda} = \left(\frac{1}{d_{n-\lambda}} L_1 x_{n-\lambda}\right)^{1/\delta}$$
$$\geq \left[\frac{D_2(n-\lambda)}{d_{n-\lambda}}\right]^{1/\delta} \left[L_2 x_{n-\lambda}\right]^{1/\delta}$$
$$\geq \left[\frac{D_2(n-\lambda)}{d_{n-\lambda}}\right]^{1/\delta} \left[L_2 x_n\right]^{1/\delta}$$
$$\frac{\Delta x_{n-\lambda}}{x_{n-\lambda}} \geq \left[\frac{D_2(n-\lambda)}{\rho_n d_{n-\lambda}}\right]^{1/\delta} \frac{\rho_n^{1/\delta}(L_2 x_n)^{1/\delta}}{x_{n-\lambda}^{\beta/\delta}} x_{n-\lambda}^{\beta/\delta-1}$$
$$= \left[\frac{D_2(n-\lambda)}{\rho_n d_{n-\lambda}}\right]^{1/\delta} w_n^{1/\delta} x_{n-\lambda}^{\beta/\delta-1}$$

and (3.5) implies,

$$\Delta w_n \leq -k\rho_n q_n - \beta \frac{\rho_n w_{n+1}^{1+1/\delta}}{\rho_{n+1}} \left[\frac{D_2(n-\lambda)}{\rho_{n+1} d_{n-\lambda}}\right]^{1/\delta} x_{n-\lambda}^{\beta/\delta-1} + w_{n+1} A_n \tag{3.6}$$

It follows from $L_3 x_n < 0$ and $0 < L_2 x_n \leq L_2 x_{n_1} = c_1 \text{ for } n \geq n_1$. Hence $c_n \Delta(L_1 x_n) = L_2 x_n \leq c_1$ for all $n \geq n_1$ and thus we have for all $n \geq n_2 = n_1 + 1$ that

$$d_n(\Delta x_n)^\delta = L_1 x_n = L_1 x_{n_1} + \sum_{s=n_1}^{n-1} \Delta(L_1 x_s)$$
$$\leq L_1 x_{n_1} + c_1 \sum_{s=n_1}^{n-1} \frac{1}{c_s}$$
$$= L_1 x_{n_1} + c_1 D_2(n)$$
$$= \left[\frac{L_1 x_{n_1}}{D_2(n)} + c_1\right] D_2(n)$$
$$\leq \left[\frac{L_1 x_{n_1}}{D_2(n)} + c_1\right] D_2(n)$$
$$= \tilde{c}_1 D_2(n)$$

where $\tilde{c}_1 = c_1 + \frac{L_1 x_{n_1}}{D_2(n)}$,

Choose $n_2 \leq n$ we have,

$$x_n = x_{n_2} + \sum_{s=n_2}^{n-1} \Delta x_s$$
$$\leq x_{n_2} + \sum_{s=n_2}^{n-1} \left(\frac{\tilde{c}_1 D_2(s)}{d_s}\right)^{1/\delta}$$
$$\leq x_{n_2} + \sum_{s=n_1}^{n-1} \left(\frac{\tilde{c}_1 D_2(s)}{d_s}\right)^{1/\delta}$$
$$= x_{n_2} + \tilde{c}_1^{1/\delta} D^*(n)$$
$$= \left[\frac{x_{n_2}}{D^*(n)} + \tilde{c}_1^{1/\delta}\right] D^*(n)$$
$$\leq \left[\frac{x_{n_2}}{D^*(n)} + \tilde{c}_1^{1/\delta}\right] D^*(n)$$
$$= c_2 D^*(n)$$

where $c_2 = \frac{x_{n_2}}{D^*(n)} + \tilde{c}_1^{1/\delta}$

Thus we have

$$x_{n-\lambda}^{\beta/\delta-1} \geq c_2^{\beta/\delta-1}\left[D^*(n-\lambda)\right]^{\beta/\delta-1} \qquad for \qquad n \geq n_2 \tag{3.7}$$

From (3.4) and (2.4) we get

$$\begin{aligned} w_n &= \frac{\rho_n L_2 x_n}{x_{n-\lambda}^{\beta}} \\ &\leq \frac{\rho_n L_2 x_{n-\lambda}}{x_{n-\lambda}^{\beta}} \\ &\leq \rho_n (D^*(n-\lambda))^{-\delta} x_{n-\lambda}^{\delta-\beta} \quad for \quad n \geq n_1 \end{aligned} \tag{3.8}$$

By using (3.7) and (3.8), we obtain

$$w_n \leq c_2^{\delta-\beta}\rho_n\left[D^*(n-\lambda)\right]^{-\beta}$$

Hence

$$w_n^{1/\delta-1} \geq c_2^{(\delta-\beta)(1/\delta-1)}\rho_n^{1/\delta-1}\left[D^*(n-\lambda)\right]^{-\beta(1/\delta-1)} \qquad for \qquad n \geq n_2 \tag{3.9}$$

By using (3.7) and (3.9) in (3.6), we obtain

$$\begin{aligned} \Delta w_n &\leq -k\rho_n q_n + A_n w_{n+1} - \frac{\beta\rho_n c_2^{\beta-\delta}}{\rho_{n+1}^2}\left[\frac{D_2(n-\lambda)}{d_{n-\lambda}}\right]^{1/\delta}\left[D^*(n-\lambda)\right]^{\beta-1} w_{n+1}^2 \\ &\leq -k\rho_n q_n + A_n w_{n+1} - B_n w_{n+1}^2 \\ &\leq -k\rho_n q_n + \frac{A_n^2}{4B_n} \end{aligned} \tag{3.10}$$

for $n \geq n_2$ where A and B are in (3.2) with $c^* = \beta c_2^{\beta-\delta}$. Summing (3.10) from n_2 to $n-1$, we see that

$$\sum_{s=n_2}^{n-1}\left[k\rho_s q_s - \frac{A_s^2}{4B_s}\right] \leq w_{n_2} - w_n \leq w_{n_2}$$

which contradicts (3.1) Next, we assume $L_1 x_n < 0$ on $[n_1, \infty)$. Then the case $L_2 x_n \leq 0$ cannot hold for all large *n*, say $n \geq n_2 \geq n_1$. Definition from $L_1 x_n$, we obtain,

$$\Delta x_n = \left(\frac{L_1 x_n}{d_n}\right)^{1/\delta} \leq \left(\frac{L_1 x_{n_2}}{d_n}\right)^{1/\delta}, n \geq n_2$$

and from (2.1) that $x_n < 0$ for all large n, which is a contradiction. Thus, assume $x_n > 0, L_1 x_n < 0$ and $L_2 x_n \geq 0$ for all large n, say $n \geq n_3 \geq n_2$. Now $v \geq u \geq n_3$, we have

$$x_u - x_v = -\sum_{s=u}^{v-1} d_\tau^{-1/\delta}\left(d_\tau(\Delta x_\tau)^\delta\right)^{1/\delta}$$
$$\geq \left(\sum_{s=u}^{v-1} d_\tau^{-1/\delta}\right)\left(-L_1 x_v\right)^{1/\delta}$$

Setting $u = n-\lambda$ and $v = n-l$, we get

$$x_{n-\lambda} \geq D_1(n-l)(-L_1 x_{n-\lambda})^{1/\delta} = D_1(n-l)x_{n-l}$$

For $n \geq n_3$ where $x_n = (-L_1 x_n)^{1/\delta} > 0$ for $n \geq n_3$. From (1.1), the fact that $\{x_n\}$is decreasing and $n-\lambda \leq n-l \leq n$, we obtain

$$\Delta(c_n \Delta z_n) + \frac{p_n}{d_n} z_{n-l} \geq k q_n [D_1(n-l)]^\beta z_{n-l}(z_{n-l})^{\beta/\delta-1}$$

Where $z = x^\delta$, since z is decreasing and $\delta \geq \beta$ there exists a constant $c_4 > 0$ such that $z_n^{\beta/\delta-1} \geq c_4$ for $n \geq n_2$. Thus

$$\Delta(c_n \Delta z_n) \geq \left[c_4 k q_n (D_1(n-l))^\beta - \frac{p_n}{d_n}\right] z_{n-l}$$

Proceeding exactly as in the proof of lemma 2.4 and 2.5, we arrive at the desired conclusion thus completing the proof.

Corollary 3.2. Assume (2.1), (2.2) and $\delta \geq \beta$. Suppose (1.2) is non-oscillatory and $A_n \leq 0$. Where A_n is defined as in (3.2). If there exist a positive sequence $\{\rho_n\}$ such that $\rho_n > 0$ and $n-\lambda \leq n-l \leq n$ for all $n \geq n_0$ and $\limsup_{n\to\infty} \sum_{s=n_1}^{\infty} \rho_s q_s = \infty$ for any $n_1 \in I$ and (2.6) or (2.11) holds with Q as in Theorem 3.1, then every solution x_n of (1.1) or $L_2 x_n$ is oscillatory.

Theorem 3.3 Let the conditions (2.1), (2.2) are holds and $\delta \geq \beta$. Suppose (1.2) is non-oscillatory. Assume that $n-\lambda \leq n-l \leq n$ for all $n \geq n_0$ and (2.6) or (2.11) holds with Q as in Theorem 3.1. If every solution of the first order delay equation

$$\Delta w_n + P_n w_{n-\lambda} + Q_{n_1} w_{n-\lambda}^{\beta/\delta} = 0 \; for\; all\; n \geq n_2 \qquad (3.11)$$

is oscillatory, then every solution $\{x_n\}$ of (1.1) or $L_2 x_n$ is oscillatory.

Proof: Let $\{x_n\}$be a non-oscillatory solution of (1.1) on $[n_1, \infty), n_1 \geq n_0$. Without loss of generality, we may assume that $x_n > 0$ and $x_{n-\lambda} > 0$ for $n \geq n_1$. From lemma 2.2, we have $L_1 x_n < 0$ or $L_1 x_n > 0$ for $n \geq n_1$. If $L_1 x_n > 0$ on $[n_1, \infty)$ then as in the proof of Theorem 3.1, we get $L_2 x_n > 0$ on $[n_1, \infty)$.

We can choose $n_2 \geq n_1$ such that $n - \lambda \geq n_1 \, for all \, n \geq n_2$, and so (2.4) gives

$$x_{n-\lambda} \geq D^*(n-\lambda)(L_2 x_{n-\lambda})^{1/\delta} \; for\, all \, n \geq n_2 \tag{3.12}$$

Using (2.3) and (3.12) in (1.1) we obtain

$$\Delta(L_2 x_n) + \frac{p_n}{d_n} D_2(n) L_2 x_n + k q_n (D^*(n-\lambda))^\beta (L_2 x_{n-\lambda})^{\beta/\delta} \leq 0$$

for $n \geq n_2$, It follows that

$$\Delta w_n + P_n w_{n-\lambda} + Q_{n_1} w_{n-\lambda}^{\beta/\delta} \leq 0 \; for\, all \, n \geq n_2,$$

where $w_n = L_2 x_n$, $P_n = \frac{p_n}{d_n} D_2(n)$, $Q_{n_1} = k q_n (D^*(n-\lambda))^\beta$

This inequality has a positive solution, and by Lemma 2.7 in [13]. We see that (3.11) has a positive solution, which is a contradiction. The case when $L_1 x_n < 0$ on $[n_1, \infty)$ is similar to that of Theorem 3.1 and hence is omitted. This completes the proof.

Corollary 3.4 Let the conditions (2.1), (2.2) are holds and $\delta \geq \beta$ suppose (1.2) is non-oscillatory. Assume that $n-\lambda \leq n-l \leq n$ for all $n \geq n_0$ and (2.6) or (2.11) holds with Q as in Theorem 3.1. If

$$\lim_{n\to\infty} \inf \sum_{s=n-\lambda}^{n} Q_{s_1} > \left(\frac{\lambda}{\lambda+1}\right)^{\lambda+1}$$

then every solution $\{x_n\}$ of (1.1) or $L_2 x_n$ is oscillatory

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4. Oscillation Results:

In this section, we establish new oscillation results for (1.1) by using double sequence.

Let us introduce a double sequence $\{H_{n,s}\}, n, s \in N(n_0)$ such that

(i) $H_{n,n} = 0 \; for\, n \in N(n_0)$

(ii) $H_{n,s} > 0 \; for\, n > s \in N(n_0)$

(iii) $\Delta_2 H_{n,s} = H_{n,s+1} - H_{n,s} \leq 0 \; for\, n > s \in N(n_0)$

Suppose that $\{h_{n,s} \mid n > s \in N(n_0)\}$ is a double sequence with $\Delta_2 H_{n,s} = -h_{n,s}\sqrt{H_{n,s}} \; for\, n > s \in N(n_0)$.

Theorem 4.1 Let the conditions (2.1), (2.2) are holds and $\delta \geq \beta$. Suppose (1.2) is non-oscillatory. Assume that there exists a positive sequence $\{\rho_n\}$ such that $\rho_n > 0$ and $n-\lambda \leq n-l \leq n,$ for all $n \geq n_0$ satisfying

$$\lim_{n\to\infty} \sup \frac{1}{H_{n,n_1}} \sum_{s=n_1}^{n-1} k\rho_s q_s H_{n,s} - \frac{P_{n,s}^2}{4B_s} = \infty \tag{4.1}$$

for all large $n \geq n_1$, where $P_{n,s} = h_{n,s} - A_s\sqrt{H_{n,s}}$,

with A and B defined as in Theorem 3.1. If (2.6) or (2.11) holds with Q as in Theorem 3.1, then every solution $\{x_n\}$ of (1.1) or $L_2 x_n$ is oscillatory.

Proof: Let $\{x_n\}$ be a non-oscillatory solution of (1.1) on $[n_1,\infty), n_1 \geq n_0$. Without loss of generality, we may assume $x_n > 0 \text{ and } x_{n-\lambda} > 0$ for $n \geq n_1$ From the Proof of Theorem (3.1),

$$\Delta w_n \leq -k\rho_n q_n + A_n w_{n+1} - B_n w_{n+1}^2$$

$$\sum_{s=n_1}^{n-1} k\rho_s q_s H_{n,s} \leq \sum_{s=n_1}^{n-1} H_{n,s}\left[-\Delta w_s + A_s w_{s+1} - B_s w_{s+1}^2\right]$$

$$= w_{n_1} H_{n,n_1} + \sum_{s=n_1}^{n-1}\left[w_{s+1}\Delta_2 H_{n,s} + H_{n,s}A_s w_{s+1} - H_{n,s}B_s w_{s+1}^2\right]$$

$$= w_{n_1} H_{n,n_1} + \sum_{s=n_1}^{n-1}\left[w_{s+1}\left(-h_{n,s}\sqrt{H_{n,s}}\right) + H_{n,s}A_s w_{s+1} - H_{n,s}B_s w_{s+1}^2\right]$$

$$= w_{n_1} H_{n,n_1} + \sum_{s=n_1}^{n-1}\left\{\left(-H_{n,s}B_s w_{s+1}^2\right) + w_{s+1}\left[\left(-h_{n,s}\sqrt{H_{n,s}}\right) + H_{n,s}A_s\right]\right\}$$

$$= w_{n_1} H_{n,n_1} - \sum_{s=n_1}^{n-1}\left\{H_{n,s}B_s w_{s+1}^2 + w_{s+1}\left[h_{n,s}\sqrt{H_{n,s}} - H_{n,s}A_s\right]\right\}$$

$$= w_{n_1} H_{n,n_1} - \sum_{s=n_1}^{n-1}\left\{H_{n,s}B_s w_{s+1}^2 + w_{s+1}\left[h_{n,s} - \sqrt{H_{n,s}}A_s\right]\sqrt{H_{n,s}}\right\}$$

$$= w_{n_1} H_{n,n_1} - \sum_{s=n_1}^{n-1}\left\{H_{n,s}B_s w_{s+1}^2 + w_{s+1}P_{n,s}\sqrt{H_{n,s}}\right\}$$

we obtain

$$= w_{n_1} H_{n,n_1} - \sum_{s=n_1}^{n-1}\left[\sqrt{H_{n,s}}\sqrt{B_s}\,w_{s+1} + \frac{P_{n,s}}{2\sqrt{B_s}}\right]^2 + \sum_{s=n_1}^{n-1}\frac{P_{n,s}^2}{4B_s}$$

$$= w_{n_1} H_{n,n_1} + \sum_{s=n_1}^{n-1}\frac{P_{n,s}^2}{4B_s}$$

Thus we obtain,

$$\frac{1}{\mathrm{H}_{\mathrm{n,n_1}}}\left[\sum_{s=n_1}^{n-1} k\rho_s q_s H_{n,s} - \frac{P_{n,s}^2}{4B_s}\right] \leq w_{n_1}$$

which is contradicts to (4.1).

Theorem 4.2 Assume all the conditions of Theorem 4.1 are hold except (4.1). Moreover, suppose that for every $n_1 > n_0$,

$$0 < \inf_{s \geq n_1}\left[\liminf_{n\to\infty}\frac{H_{n,s}}{H_{n,n_1}}\right] \leq \infty,$$

$$\limsup_{n\to\infty}\frac{1}{H_{n,n_1}}\sum_{s=n_1}^{n-1}\frac{c_3 P_{n,s}^2}{B_s} < \infty,$$

and there is a sequence $\{\chi_n\}$ such that

$$\sum_{s=n_1}^{\infty}\frac{1}{c_3}B_s\left[\chi_s^+\right]^2=\infty \text{ where } \chi_s^+=\max\{\chi_s,0\}$$

and

$$\limsup_{n\to\infty}\frac{1}{H_{n,n_1}}\sum_{s=n_1}^{n-1}\left[k\rho_s q_s H_{n,s}-\frac{p_{n,s}^2}{4B_s}\right]\geq\chi_{n_1} \qquad (4.2)$$

then every solution $\{x_n\}$ of (1.1) or L_2x_n is oscillatory.

Proof: Let $\{x_n\}$ be a non-oscillatory solution of (1.1) on $[n_1,\infty)$. Without loss of generality, we may assume $x_n>0 \text{ and } x_{n-\lambda}>0 \text{ for } n\geq n_1$. Proceeding as in the proof of Theorem 4.1, we obtain

$$\sum_{s=n_1}^{n-1}k\rho_s q_s H_{n,s}\leq w_{n_1}H_{n,n_1}+\sum_{s=n_1}^{n-1}\frac{p_{n,s}^2}{4B_s}-\sum_{S=n_1}^{n-1}\left[\sqrt{H_{n,s}B_s}\,w_{s+1}+\frac{P_{n,s}}{2\sqrt{B_s}}\right]^2$$

Using (4.2) we obtain

$$\chi_{n_1}\leq\limsup_{n\to\infty}\frac{1}{H_{n,n_1}}\sum_{s=n_1}^{n}\left[k\rho_s q_s H_{n,s}-\frac{P_{n,s}^2}{4B_s}\right]$$

$$\leq w_{n_1}-\liminf_{n\to\infty}\frac{1}{H_{n,n_1}}\sum_{s=n_1}^{n-1}\left[\sqrt{H_{n,s}B_s}\,w_{s+1}+\frac{P_{n,s}}{2\sqrt{B_s}}\right]^2$$

and hence

$$\liminf_{n\to\infty}\frac{1}{H_{n,n_1}}\sum_{s=n_1}^{n-1}\left[\sqrt{H_{n,s}B_s}\,w_{s+1}+\frac{P_{n,s}}{2\sqrt{B_s}}\right]^2<\infty \qquad (4.3)$$

Define

$$c_{n_1}=\frac{1}{H_{n,n_1}}\sum_{s=n_1}^{n-1}H_{n,s}B_s w_{s+1}^2$$

$$c_{n_2}=\frac{1}{H_{n,n_1}}\sum_{s=n_1}^{n-1}\sqrt{H_{n,s}}\,P_{n,s}w_{s+1}$$

It follows from (4.3) that

$$\liminf_{n\to\infty}[c_{n_1}+c_{n_2}]<\infty$$

The remainder of the proof is similar to that of [9] and hence it is omitted. The rest of the proof of the case if $x_n>0 \text{ and } L_1x_n<0$ is similar to that of the proof of Theorem 3.1 and hence it is omitted.

5.Examples:

In this section, we present some examples.

Example 5.1 Consider the third order non linear damped delay difference equation of the form,

$$\Delta\left(\frac{1}{2}\Delta(\Delta x_n)^3\right)+2(\Delta x_n)^3+32g(x_{n-2})=0 \qquad n\geq n_0 \tag{5.1}$$

Here $c_n=\frac{1}{2},\ d_n=1,\ \delta=3,\ \ p_n=2, k=1,\ \ \beta=3, and \qquad q_n=32$

All the conditions of corollary 3.2 are satisfied with $\rho_n=n$. In fact $x_n=(-1)^n$ is one such oscillatory solution of equation (5.1)

Example 5.2 Consider the third order non linear damped delay difference equation of the form,

$$\Delta\left(2\Delta(\Delta x_n)^3\right)+(\Delta x_n)^3+\frac{72}{3}g(x_{n-4})=0 \qquad n\geq 4 \tag{5.2}$$

Here $c_n=2,\ d_n=1,\ \delta=3,\ \ p_n=1,\ k=3,\ \ \beta=1, and \qquad q_n=\frac{72}{3}$

All the conditions of corollary 3.2 are satisfied. In fact $x_n=(-1)^{n+1}$ is one such oscillatory solution of equation (5.2)

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