UNIQUENESS OF SOLUTION FOR BOUNDARY VALUE PROBLEMS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

This paper mainly deals with the exclusivity of nonlinear boundary value problems. The problems are scrutinized for fractional integro-differential equations. The major focal point of this study is to analyse the Lipschitz constant, which is related to the first eigen values and it also aims to analyse the u_0 – positive operator

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1 Introduction

Fractional calculus is concerned with the generalization of integrals and derivatives of noninteger.Fractional derivatives grants an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the keybenefit of fractional differential equations in similarity with classical integer-order models. Fractional differential equationcome up in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology and so additionally, involves derivatives of fractional order.

The theory and application of integrodifferential equations plays ansignificant role in the mathematical modeling of many fields: physical, biological phenomena and engineering sciences in which it is necessary to take into account the effect of real world problems. Most of the practical systems are integrodifferential equations in nature and hence the study of integrodifferential equations becomes very important.

In recent years, many researchers have studied about the boundary value problems of nonlinear fractional differential equations [1, 2, 4, 6, 7, 8, 10-14]. Most of the resultshave explained that the fractional differential equations had as a minimum single and multiple positive solutions by using techniques of nonlinear analysis. For instance, the authors[10] considered the existence of multiple positive solutions for the following fractional differential equation with a negatively perturbed term

$$\begin{cases} -D_{0+}^{p}\, x(t) = p(t)f\big(t, x(t)\big) - q(t), \quad t \in (0,1) \\ \qquad x(0) = x'(0) = 0, \quad x(1) = 0, \end{cases}$$

whereD_{0+}^{p} is the standard Riemann-Liouville derivative, $2 < p \leq 3$ is a real number, $q: (0,1) \to [0, \infty)$is lebesgue integrable and does not vanish identically on any subinterval of (0,1). They recognized the existence results by Krasnosel'skii's fixed point theorem in a cone.

However, few results can be found in the literature concerning the uniqueness of solutions for boundary value problems of fractional differential equations [10, 11, 12, 13, 14].

Motivated by [9], we consider the following boundary value problems for fractional integro differential equations to develop new uniqueness results.

$$\begin{cases} D^p y(t) + p(t) f\left(t, y(t), \int_0^t h\big(t, s, y(s)\big) ds\right) + q(t) = 0 \\ y(0) = y'(0) = 0, \quad y(1) = 0, \end{cases} \tag{1.1}$$

where$2 < p \leq 3$ is a real number. Under the assumption that f is a Lipchitz continuous function, by use of $u_0 -$ positive operator, we study the uniqueness existence of solution for the fractional integro-differential equation(1.1). The interesting point is that the Lipschitz constant is related to the first eigenvalues corresponding to the relevant operators.

The rest of this paper is organized as follows. In section 2, some preliminaries are offered. In section 3, we study the uniqueness existence of solution for the fractional integrodifferentialequation(1.1).

2 Preliminaries

For the expediency of the reader, we present here some necessary definitions from fractional calculus theory. These definitions and properties can be originated in the recent monograph [4, 6, 7, 8].

Definition 2.1. The Riemann-Liouville fractional integral of order $p > 0$ of a function $f: (0, \infty) \to \mathbb{R}$ is given by

$$I^p f(t) = \frac{1}{\Gamma(p)} = \int_0^t (t-s)^{p-1} f(s) ds,$$

Provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $p > 0$ of a continuous function $f: (0, \infty) \to \mathbb{R}$ is given by

$$D^p f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{p-n+1}} ds,$$

where $n - 1 \leq \alpha < n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

In Banach space $B = C[0,1]$ in which the norm is defined by $\|x\| = max_{t \in [0,1]} |x(t)|$, we set $P = \{x \in C[0,1] | x(t) \geq 0, \ \forall \ t \in [0,1]\}$. P is a positive cone in $C[0,1]$. Throught this paper, the partial ordering is always given by P.

The following notion is the due to krasnosel'skii [5].

Definition 2.3. We say that a bounded linear operator $E: B \to B$ is $u_0 -$ positive on the cone P if there exists $u_0 \in P\backslash\{\theta\}$ such that for every $x \in P\backslash\{\theta\}$ there exist a natural number n and positive constants $\alpha(x),\ \beta(x)$ such that

$$\alpha(x)u_0 \leq E^n x \leq \beta(x)u_0.$$

$\psi^* \in B$ be called a positive eigenfunction of linear operator E if $\psi^* \in P\backslash\{\theta\}$ and there exists $\lambda > 0$ such that $\lambda E\psi^* = \psi^*$.

The following is an existence and the uniqueness results of solution for a linear boundary value problem, which is important for us in the following analysis.

Lemma 2.1. Let $\alpha \in \mathbb{R}$, $\sigma \in C(0,1) \cap L(0,1)$ and $2 < p \leq 3$, then the unique solution of

$$\begin{cases} D^p x(t) + \sigma(t) = 0, & t \in (0,1), \\ x(0) = x'(0) = 0, & x(1) = a, \end{cases}$$

isgiven by

$$x(t) = at^{p-1} + \int_0^1 G(t,s)\,\sigma(s)ds,$$

Where $G(t,s)$ is Green's function given by

$$G(t,s) = \begin{cases} \frac{(1-s)^{p-1}t^{p-1} - (t-s)^{p-1}}{\Gamma(p)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{p-1}t^{p-1}}{\Gamma(p)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.1}$$

Proof: The proof is alike to the argument of Lemma 2.2 in [10]. So we omit it.

Lemma 2.2.([10]). The function $G(t,s)$ defined by (2.1) satisfies the following conditions:

(1) $t^{p-1}(1-t)s(1-s)^{p-1} \leq \Gamma(p)G(t,s) \leq (p-1)s(1-s)^{p-1},\ t,s \in (0,1)$.

(2) $t^{p-1}(1-t)s(1-s)^{p-1} \leq \Gamma(p)G(t,s) \leq (p-1)t^{p-1}(1-t),\ t,s \in (0,1)$.

Let the operators E and F be defined by

$$(Ex)(t) = \int_0^1 G(t,s)p(s)x(s)ds,\ \ t \in [0,1],\ \ x \in C[0,1].$$

$$(Fx)(t) = \int_0^1 G(t,s)\big[p(s)f\big(s,x(s)\big) + q(s)\big]ds,\ \ t \in [0,1],\ \ x \in C[0,1]$$

respectively. It is not to difficult to verify that $E: B \to B$ is linear completely continuous and $E(P) \subset P$.

Lemma 2.3. E is u_0 – positive operator with $u_0(t) = t^{p-1}(1-t)$.

Proof: For any $x \in P\backslash\{\theta\}$, it follows from Lemma 2.2 that

$$(Ex)(t) = \int_0^1 G(t,s)p(s)x(s)ds \le \frac{p-1}{\Gamma(p)}\int_0^1 p(s)x(s)ds.\ t^{p-1}(1-t)$$

On the other hand, by Lemma 2.2 again, we have

$$(Ex)(t) = \int_0^1 G(t,s)p(s)x(s)ds \ge \frac{1}{\Gamma(p)}\int_0^1 s(1-s)^{p-1}p(s)x(s)ds.\,t^{p-1}(1-t).$$

The inequalities thus obtained mean that E is u_0 - positive operator with $u_0(t) = t^{p-1}(1-t)$. This completes the evidence.

Lemma 2.4. ([3],[5]). Suppose that $E: B \to B$ is a completely continuous linear operator and $E(P) \subset P$. If there exist $\phi \in B\backslash(-P)$ and a constant $c > 0$ such that $cE\phi \ge \phi$, then the spectral radius $r(E) \ne 0$ and E has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = (r(E))^{-1}$, i.e. $\psi = \lambda_1 E\psi$.

It follows from Lemmas 2.3 and 2.4 that the spectral radius $r(E) \ne 0$ and E has a positive eigenfunction $\psi^*(t)$ corresponding to its first eigenvalue $\lambda_1 = (r(E))^{-1}$.

Remark 2.1. Let ψ^* be the positive eigenfunction of E corresponding to λ_1, thus $\lambda_1 E\psi^* = \psi^*$. Then by Lemma 2.3 and Definition 2.3, there exist $r_1(\psi^*), r_2(\psi^*) > 0$ such that

$$r_1(\psi^*)u_0 \le E\psi^* = \frac{1}{\lambda_1}\psi^* \le r_2(\psi^*)u_0.$$

Hence we obtained that E is ψ^*- positive operator.

3 Main results

We require the following hypothesis to prove our core results.
Let λ_1 be the first eigenvalue of E.
(H_1) $p : (0,1) \to [0,+\infty)$is continuous and does not disappear identically on any subinterval of (0,1) such that

$$0 < \int_0^1 p(s)ds < +\infty$$

(H_2) $f: [0,1] \times \mathbb{R} \times B \to \mathbb{R}$ and there exist two positive constants k_1 , k_2 such that

$$|f(t,x,\overline{x})-f(t,y,\overline{y})|\leq \lambda_1[k_1|x-y|+k_2|\overline{x}-\overline{y}|]$$

(H_3) $h:\mathbb{R}\times\mathbb{R}\times \mathrm{B}\to\mathbb{R}$ is continuous and there exist a positive constant k such that

$$|h(t,s,x)-h(t,s,y)|\leq k|x-y|$$

(H_4) $q:(0,1)\to\mathbb{R}$ is continuous and Lebesgueintegrable.

Theorem 3.1. Assume that (H_1) *to* (H_4) holds. Then (1.1) has a unique solution y^* in B, and for any $y_0\in B$, the iterative sequence $y_n=Fy_{n-1}(n=1,2,\dots)$ converges to y^*.

Proof: It is easy to prove that $F:P\to P$ is completely continuous and the survival of solution for (1.1) is equivalent to that of fixed point of F in B.
For any given $y_0\in B$
Let $y_n=Fy_{n-1}(n=1,2,\dots)$
By Lemma 2.3 and Remark 2.1, there exist $\gamma=\gamma(|y_1-y_0|)>0$ such that
$E(|y_1-y_0|)(t)\leq\gamma\psi^*(t), t\in[0,1]$.
For $m\in N$,
$|y_{m+1}(t)-y_m(t)|=|(Fy_m)(t)-(Fy_{m-1})(t)|$

$$=\left|\int_0^1 G(t,s)[p(s)f(s,y_m(s),\int_0^t h(t,s,y_m(s))ds)]ds - \int_0^1 G(t,s)[p(s)f(s,y_{m-1}(s),\int_0^t h(t,s,y_{m-1}(s))ds)]ds\right|$$

$$\leq \int_0^1 G(t,s)\left|p(s)f\left(s,y_m(s),\int_0^t h(t,s,y_m(s))ds\right) - p(s)f\left(s,y_{m-1}(s),\int_0^t h(t,s,y_{m-1}(s))ds\right)ds\right|$$

$$\leq \lambda_1(k_1+kk_2)E(|y_m-y_{m-1}|)(t)\leq\lambda_1\overline{k}E(|y_m-y_{m-1}|)(t)\quad where\ \overline{k}=k_1+kk_2$$

$$\leq\cdots\leq\overline{k}^m\lambda_1^mE^m(|y_1-y_0|)(t)\leq\overline{k}^m\lambda_1^mE^{m-1}(\gamma\psi^*(t))=\overline{k}^m\lambda_1^m\gamma E^{m-1}(\psi^*(t))$$

$$=\overline{k}^m\gamma\lambda_1\psi^*(t)$$

Thus for $n,m\in N$,
$|y_{n+m+1}(t)-y_n(t)|=|y_{n+m+1}(t)-y_{n+m}(t)+\cdots y_{n+1}(t)-y_n(t)|$

$$\leq|y_{n+m+1}(t)-y_{n+m}(t)|+\cdots+|y_{n+1}(t)-y_n(t)|$$

$$\leq\gamma\lambda_1\left[(\overline{k})^{n+m}+\cdots(\overline{k})^n\right]\psi^*(t)=\gamma\lambda_1\frac{(\overline{k})^n\left(1-(\overline{k})^{m+1}\right)}{1-k}\psi^*(t)$$

Therefore $\|y_{n+m+1} - y_{n+m}\| \le \gamma\lambda_1 \frac{(\overline{k})^n\left(1-(\overline{k})^{m+1}\right)}{1-k} \|\psi^*\| \to 0 \quad as \;\; n,m \;\to \infty$

By the completeness of B, there exist a $y^* \in B$ such that $\lim_{n\to\infty} y_n = y^*$. Passing to the limit into $y_{n+1} = Fy_n$ and using the fact that F is continuous, it follows that y^* is a fixed point of F in B.

Next we show that F has atmost one fixed point in B. Suppose there exist two elements $x, y \in B$ with $x = Fx \; and \; y = Fy$. By Lemma 2.3, there exist $\gamma = \gamma(|x-y|) > 0$ such that

$$E(|x-y|)(t) \le \gamma\psi^*(t), \quad t \in [0,1]$$

Then for all $n \in N$, we have

$$|x(t) - y(t)| = |(F^n x)(t) - (F^n y)(t)| \le \left(\overline{k}\right)^n \gamma\lambda_1\psi^*(t).$$

which can happen only if $x = y$. This implies that F has atmost one fixed point. Therefore y^* is the unique fixed point of F in B. This completes the proof.

Theorem 3.2. Suppose that there exist $y_0 \in B$, $\overline{k} \in [0,1)$ satisfies the following conditions :

$$D^p y_0(t) + p(t) f\left(t, y_0(t), \int_0^t h\big(t,s,y_0(s)\big)ds\right) + q(t) \ge 0\,, \quad t \in (0,1)$$

$$y_0(0) = y_0'(0) = 0\,;\; y_0(1) \le 0$$

$$0 \le f\big(t, u(t), \overline{u}(t)\big) - f\big(t, v(t), \overline{v}(t)\big) \le \overline{k}\lambda_1\left[\big(u(t) - v(t)\big) + \big(\overline{u}(t) - \overline{v}(t)\big)\right],$$

$$u(t) \ge v(t); \overline{u}(t) \ge \overline{v}(t) \;\; \forall\, t \in [0,1] and\; u, v, \overline{u}, \overline{v} \in \Omega$$

in which $\Omega = \{y \in B / y \ge y_0\}$. Then (1.1) has a unique solution $y^* \; in\, \Omega$.

Proof: It follows from Lemma 2.1 that F is increasing on D and $y_0 \le Fy_0$, so we obtain $F(D) \subset D$.

Let $y_n = Fy_{n-1} (n = 1,2,3 \ldots)$, then we have $y_0 \le y_1 \le \cdots \le y_n \le \cdots$

By Lemma 2.3, there exist $\gamma > 0$ such that

$$E(y_1 - y_0) \le \gamma\psi^*$$

Then for $\forall\, n \in N \; and \; t \in [0,1]$, we have

$$0 \le y_{n+1}(t) - y_n(t) = Fy_n(t) - Fy_{n-1}(t) \le \overline{k}\lambda_1 E(y_n - y_{n-1})(t)$$
$$\le \overline{k}\lambda_1 E(Fy_{n-1} - Fy_{n-2})(t) \ldots \le \left(\overline{k}\lambda_1 E\right)^n (y_1 - y_0)(t) \le \gamma\left(\overline{k}\right)^n \lambda_1\psi^*(t)$$

Thus for $n, m \in N$,

$$|y_{n+m}(t) - y_n(t)| = |y_{n+m}(t) - y_{n+m-1}(t) + \cdots + y_{n+1}(t) - y_n(t)|$$

$$\le |y_{n+m}(t) - y_{n+m-1}(t)| + \cdots + |y_{n+1}(t) - y_n(t)|$$

$$\le \gamma\lambda_1\left[\left(\overline{k}\right)^{n+m-1} + \cdots \left(\overline{k}\right)^n\right]\psi^*(t) = \gamma\lambda_1 \frac{\left(\overline{k}\right)^n\left(1 - \left(\overline{k}\right)^m\right)}{1 - \overline{k}}\psi^*(t)$$

This shows that

$$\|y_{n+m} - y_n\| \leq \gamma\lambda_1 \frac{(\overline{k})^n \left(1 - (\overline{k})^m\right)}{1 - \overline{k}} \|\psi^*(t)\|$$

So, $\{y_n\}$ is a Cauchy sequence in B and since that B is a Banach space there exist $y^* \in B$ such that $\lim_{n\to\infty} y_n = y^*$. Hence y^* is a fixed point of F in Ω.

In the following we will show that y^* is the unique fixed point of F in Ω. Suppose that there exist aelement $y \in D \; with \; y = Fy$.

By Remark 2.1, there exist $\gamma_1 > 0$ such that $E(y_1 - y_0) \leq \gamma_1\psi^*$ and for any $n \in N, we \; have \; y \geq y_n \geq y_0$
Therefore, $y \geq y^* \geq y_n \geq y_0$. Then for all $n \in N$ and $t \in [0,1]$, we have

$$|y(t) - y^*(t)| = |y(t) - y_n(t)| + |y^*(t) - y_n(t)|$$

$$\leq |(F^n y)(t) - (F^n y_0)(t)| + |(F^n y^*)(t) - (F^n y_0)(t)|$$

$$\leq 2|(F^n y)(t) - (F^n y_0)(t)| \leq 2\gamma_1 (\overline{k})^n \lambda_1 \psi^*(t)$$

Thus, we get $y = y^*$. This completes the proof.

Similar to theorem 3.2, we also have the following results.

Theorem 3.3. Suppose that there exist $y_0 \in B, \overline{k} \in [0,1)$such that satisfies the following conditions :

$$D^p y_0(t) + p(t) f\left(t, y_0(t), \int_0^t h(t, s, y_0(s)) ds\right) + q(t) \leq 0, \quad t \in (0,1)$$

$$y_0(0) = y_0'(0) = 0 \; ; \; y_0(1) \geq 0$$

$$0 \leq f(t, u(t), \overline{u}(t)) - f(t, v(t), \overline{v}(t)) \leq \overline{k}\lambda_1[(u(t) - v(t)) + (\overline{u}(t) - \overline{v}(t))],$$

$$u(t) \geq v(t); \overline{u}(t) \geq \overline{v}(t) \;\; \forall \, t \in [0,1] and \; u, v, \overline{u}, \overline{v} \in \Omega$$

in which $\Omega = \{y \in B / y \leq y_0\}$. Then (1.1) has a unique solution $y^* \; in \; \Omega$.

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