Some Common Fixed Point Theorems for Hardy Roger Type Contraction on Partial b-Metric Spaces

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Abstract
We prove some fixed point theorems using Hardy Roger type contraction in the setting of b-metric as well as partial b-metric spaces in order to find the existence and uniqueness of the common fixed point. We also provide examples to illustrate the existence of fixed point and its uniqueness.

Keywords: Common fixed point, Hardy roger type contraction, Partial b-metric

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1. Introduction
Fixed point theory is the most important and unique instrument in the field of Science, Engineering and Technological World. The method of fixed point theory is used in analysis from 20th Century onwards. It was introduced by Joseph Liouville in 1837 and by Charles Emile Picard in 1890 based on the method of successive approximations and it is relevant in finding the existence of solutions in differential equations.

The pioneering work of Classical Theory was given by Stephan Banach which was established in 1922. In point of the historical view, there are some Mathematicians who completed the results in Fixed Point Theory, they are L.E.T. Brower, W.A. Kirk, Sĩlms, Granas and Dugundiji.

The concept of b – metric space was introduced by Bakhtin in 1989. Further it was worked out and expanded by Czerwik in 1993. Making use of their results as better tools, many scholars derived some renowned Banach fixed point theorems in the b - metric spaces and partial b-metric spaces. The partial b-metric was introduced by O’Neill and it is also known as dualistic partial metric space.
2. Preliminaries

Definition 2.1: [11]
Let $\chi_b$ be a non-empty set and $q_b \geq 1$ be a given real number and also the mapping $d_{b_m}: \chi_b \times \chi_b \rightarrow R^+$ ($R^+$ stands for non-negative real) when it satisfies following conditions ,

For all $g_b, i_b, k_b \in \chi_b$,
- $d_{b_m}(g_b, i_b) = 0$ if and only if $g_b = i_b$,
- $d_{b_m}(g_b, i_b) = (i_b, g_b)$,
- $d_{b_m}(g_b, k_b) \leq q_b [d_{b_m}(g_b, i_b) + (i_b, k_b)]$

The pair $(\chi_b, d_{b_m})$ is called b- metric space.

It is the extension of usual metric space.

Definition 2.2: [8]
Let $\Phi_s$ be the non-empty set and $p_m: \Phi_s \times \Phi_s \rightarrow [0, \infty)$ be a mapping, then the following conditions are satisfied, for all $\tau, \varphi, \gamma \in \Phi_s$ :
- $\tau = \varphi \Leftrightarrow p_m(\tau, \tau) = p_m(\varphi, \varphi) = p_m(\tau, \varphi)$;
- $p_m(\tau, \tau) \leq p_m(\tau, \varphi)$;
- $p_m(\tau, \varphi) = p_m(\varphi, \tau)$;
- $p_m(\tau, \varphi) \leq p_m(\tau, \gamma) + p_m(\gamma, \varphi) - p_m(\gamma, \gamma)$

In which the pair $(\Phi_s, p_m)$ is called as partial metric space.

Definition 2.3: [12]

Let $\chi_b$ be a non-empty set and $q_b \geq 1$ be a given real number and $(\chi_b, p_{b_m})$ be a partial b-metric space when it fulfills the following conditions for all $g_b, i_b, k_b \in \chi_b$,
- $g_b = i_b$ if and only if $p_{b_m}(g_b, g_b) = p_{b_m}(g_b, i_b) = p_{b_m}(i_b, i_b)$;
- $p_{b_m}(g_b, g_b) \leq p_{b_m}(g_b, i_b)$;
- $p_{b_m}(g_b, i_b) = p_{b_m}(i_b, g_b)$;
- $p_{b_m}(g_b, i_b) \leq q_b [p_{b_m}(g_b, k_b) + p_{b_m}(k_b, i_b) - p_{b_m}(k_b, k_b)]$

This is known as partial b- metric space.

The number $q_b \geq 1$ is the coefficient of $(\chi_b, p_{b_m})$.

Definition 2.4: [12]

Let $\chi_b$ be a non-empty set and $q_b \geq 1$ be a given real number and $(\chi_b, p_{b_m}, q_b)$ be a partial b-metric space. $g_{b_n}$ be any sequence in $\chi_b$, and $g_b \in \chi_b$, then,
- The sequence $\{g_{b_n}\}$ is said to be convergent if it converges to $g_b$ if $\lim_{n \to \infty} p_{b_m}(g_{b_n}, g_b)$ exists and is finite.
- The $\{g_{b_n}\}$ sequence is said to be Cauchy sequence in $(\chi_b, p_{b_m}, q_b)$ if $\lim_{n,m \to \infty} d_{b_m}(g_{b_n}, g_{b_m})$ exists and finite.
\( (\chi_b, p_{bm}, q_b) \) is called as complete partial b-metric space if for all Cauchy Sequence \( g_{bn} \) in \( \chi_b \) and there exists \( g_b \in \chi_b \) such that,
\[
\lim_{n,m \to \infty} d_{bm}(g_{bn}, g_{bm}) = \lim_{n \to \infty} p_{bm}(g_{bn}, g_b) = p_{bm}(g_b, g_b).
\]
We should remember that limit of the convergent sequence may not be unique.

**Definition 2.5**: [5]

Given a metric space \((\Phi_s, d_b)\), the mapping \( T_{bm} : \Phi_s \to \Phi_s \) is said to be an interpolative Kannan contraction mapping if,
\[
d_b(T_{bm} \xi_m, T_{bm} \chi_m) \leq \eta_s (d_b(\xi_m, T_{bm} \xi_m))^{\tau} \cdot d_b(\chi_m, T_{bm} \xi_m)\]
\[
\leq \eta_s \left[ (d_b(\xi_m, \chi_m))^{\sigma} \cdot (d_b(\xi_m, T_{bm} \xi_m))^{\tau} \cdot (d_b(\chi_m, T_{bm} \chi_m))^{\rho} \right] \cdot \left( \frac{1}{2} \left[ d_b(\xi_m, T_{bm} \chi_m) + d_b(\chi_m, T_{bm} \xi_m) \right] \right)^{1-\tau-\sigma-\rho},
\]
\[ (1) \]

For all \( \xi_m, \chi_m \in \Phi_s \setminus \text{Fix}(T_{bm}) \).

**Main Results**

3. Fixed Point theorems with the Contraction on b-Metric Spaces and Partial b-Metric Spaces

We begin the chapter by giving the notion of

**Interpolative Hardy – Rogers type contractions.**

**Definition 3.1**

Let \((\Phi_s, d_b)\) be a metric space. We can say that the self – mapping \( T_{bm} : \Phi_s \to \Phi_s \) is an interpolative Hardy Rogers type contraction if there exists \( \eta_s \in [0,1) \) and \( \tau, \sigma, \rho \in (0,1) \) with \( \tau + \sigma + \rho < 1 \), such that,
\[
d_b(T_{bm} \xi_m, T_{bm} \chi_m) \leq \eta_s \left[ (d_b(\xi_m, \chi_m))^{\sigma} \cdot (d_b(\xi_m, T_{bm} \xi_m))^{\tau} \cdot (d_b(\chi_m, T_{bm} \chi_m))^{\rho} \right] \cdot \left( \frac{1}{2} \left[ d_b(\xi_m, T_{bm} \chi_m) + d_b(\chi_m, T_{bm} \xi_m) \right] \right)^{1-\tau-\sigma-\rho}.
\]
\[ (2) \]

For all \( \xi_m, \chi_m \in \Phi_s \setminus \text{Fix}(T_{bm}) \).

**Theorem 3.2**

Let \((\Phi_s, d_b)\) a complete b-metric space, \( q_b \geq 1 \) be a real number and \( T_{bm} \) be an interpolative Hardy Rogers type contraction. Then \( T_{bm} \) has a unique fixed point in \( \Phi_s \).
Proof:

Given,

\[ (\Phi_d, d_b) \] be a complete b-metric space and \( T_{b_m} \) be an interpolative type contraction.

To prove,

\[ T_{b_m} \text{ has a fixed point in } \Phi_d. \]

As we start from \( \zeta_{m_0} \in \Phi_d \), considering \( \{ \zeta_{m_n} \} \), given as

\[ \zeta_{m_n} = T_{b_m}^n (\zeta_{m_0}) \]

for each positive integer \( n \).

If there exists \( n \), such that \( \zeta_{m_n} = \zeta_{m_{n+1}} \), then \( \zeta_{m_n} \) is a fixed point of \( T_{b_m} \).

Let us assume that \( \zeta_{m_n} \neq \zeta_{m_{n+1}} \) for all \( n \geq 0 \).

By substituting the values \( \zeta_{m_n} = \zeta_{m_n} \) and \( \chi_m = \zeta_{m_n-1} \) in the notion of interpolative Hardy- Rogers type contraction,

\[ d_b(T_{b_m} \zeta_m, T_{b_m} \chi_m) \]

\[ \leq \eta_s \left[ (d_b(\zeta_m, \chi_m))^\sigma \cdot (d_b(\zeta_m, T_{b_m} \chi_m))^\tau \cdot (d_b(\chi_m, T_{b_m} \chi_m))^\rho \right] \]

\[ \leq \eta_s \left[ \frac{1}{2} (d_b(\zeta_m, T_{b_m} \chi_m) + d_b(\chi_m, T_{b_m} \chi_m)) \right]^{1-\tau-\sigma-\rho} \quad (3) \]

We shall consider \( \zeta_m = \zeta_{m_n} \cdot \chi_m = \zeta_{m_n-1} \), and we get,

\[ d_b(\zeta_{m_n+1}, \zeta_{m_n}) \]

\[ = d_b(T_{b_m} \zeta_m, T_{b_m} \zeta_{m_n-1}) \]

\[ \leq \eta_s \left[ d_b(\zeta_m, \zeta_{m_n-1}) \right]^\sigma \cdot \left[ d_b(\zeta_m, T_{b_m} \zeta_{m_n}) \right]^\tau \cdot \left[ d_b(\zeta_{m_n-1}, T_{b_m} \zeta_{m_n-1}) \right]^\rho \]

\[ \cdot \left[ \frac{1}{2} (d_b(\zeta_{m_n}, T_{b_m} \zeta_{m_n-1}) + d_b(\zeta_{m_n-1}, T_{b_m} \zeta_{m_n})) \right]^{1-\tau-\sigma-\rho} \]

\[ \leq \eta_s \left[ d_b(\zeta_m, \zeta_{m_n-1}) \right]^\sigma \cdot \left[ d_b(\zeta_m, \zeta_{m_n+1}) \right]^\tau \cdot \left[ d_b(\zeta_{m_n-1}, \zeta_{m_n}) \right]^\rho \]

\[ \cdot \left[ \frac{1}{2} (d_b(\zeta_{m_n}, \zeta_{m_n}) + d_b(\zeta_{m_n-1}, \zeta_{m_n+1})) \right]^{1-\tau-\sigma-\rho} \]

Since, \( d_b(\zeta_{m_n}, \zeta_{m_n}) = 0 \), we have,

\[ \leq \eta_s \left[ d_b(\zeta_m, \zeta_{m_n-1}) \right]^\sigma \cdot \left[ d_b(\zeta_m, \zeta_{m_n+1}) \right]^\tau \cdot \left[ d_b(\zeta_{m_n-1}, \zeta_{m_n}) \right]^\rho \]

\[ \cdot \left[ \frac{1}{2} (d_b(\zeta_{m_n-1}, \zeta_{m_n+1})) \right]^{1-\tau-\sigma-\rho} \]

\[ \leq \eta_s \left[ d_b(\zeta_m, \zeta_{m_n-1}) \right]^\sigma \cdot \left[ d_b(\zeta_m, \zeta_{m_n+1}) \right]^\tau \cdot \left[ d_b(\zeta_{m_n-1}, \zeta_{m_n}) \right]^\rho \]

\[ \cdot \left[ \frac{1}{2} (d_b(\zeta_{m_n-1}, \zeta_{m_n}) + d_b(\zeta_{m_n}, \zeta_{m_n+1})) \right]^{1-\tau-\sigma-\rho} \quad (3) \]

Supposing that,

\[ d_b(\zeta_{m_n-1}, \zeta_{m_n}) < d_b(\zeta_{m_n}, \zeta_{m_n+1}) \text{ for some } n \geq 1, \text{ thus we get the result as,} \]

\[ \frac{1}{2} (d_b(\zeta_{m_n-1}, \zeta_{m_n}) + d_b(\zeta_{m_n}, \zeta_{m_n+1})) \leq d_b(\zeta_{m_n}, \zeta_{m_n+1}) \]

So, the inequality (2) can be written as,
\[
\begin{align*}
\leq \eta_s \left[ d_b(\zeta_{m_n}, \zeta_{m_{n-1}}) \right]^{\sigma} \cdot \left[ d_b(\zeta_{m_n}, \zeta_{m_{n+1}}) \right]^{\tau} \cdot \left[ d_b(\zeta_{m_{n-1}}, \zeta_{m_n}) \right]^{\rho} \\
\cdot \left[ \frac{1}{2} (d_b(\zeta_{m_{n-1}}, \zeta_{m_n}) + d_b(\zeta_{m_n}, \zeta_{m_{n+1}})) \right]^{1-(\tau+\sigma+\rho)} \\
\leq \eta_s \left[ d_b(\zeta_{m_n}, \zeta_{m_{n-1}}) \right]^{\sigma} \cdot \left[ d_b(\zeta_{m_n}, \zeta_{m_{n+1}}) \right]^{\tau} \cdot \left[ d_b(\zeta_{m_{n-1}}, \zeta_{m_n}) \right]^{\rho} \\
\cdot \left[ d_b(\zeta_{m_n}, \zeta_{m_{n+1}}) \right] \cdot \left[ d_b(\zeta_{m_{n-1}}, \zeta_{m_n}) \right]^{-\tau-\sigma-\rho} \\
\left[ d_b(\zeta_{m_n}, \zeta_{m_{n+1}}) \right]^{\sigma+\rho} \leq \eta_s \left[ d_b(\zeta_{m_n}, \zeta_{m_{n-1}}) \right]^{\sigma+\rho} \quad \forall \ n \geq 1 
\end{align*}
\]

Now we can conclude that, 
\[
[d_b(\zeta_{m_{n-1}}, \zeta_{m_n})] \geq [d_b(\zeta_{m_n}, \zeta_{m_{n+1}})],
\]
This is contradiction to the assumption that we had considered already.

Hence we derive as the conclusion that, 
\[
[d_b(\zeta_{m_n}, \zeta_{m_{n+1}})] \leq [d_b(\zeta_{m_{n-1}}, \zeta_{m_n})], \quad \forall \ n \geq 1.
\]

Thus, 
\[
[d_b(\zeta_{m_{n-1}}, \zeta_{m_n})]
\]
is a non – increasing sequence with positive terms.

The set we have, 
\[
\varphi = \lim_{n \to \infty} d_b(\zeta_{m_{n-1}}, \zeta_{m_n})
\]

So, we get, 
\[
\left[ \frac{1}{2} (d_b(\zeta_{m_{n-1}}, \zeta_{m_n}) + d_b(\zeta_{m_n}, \zeta_{m_{n+1}})) \right] \leq d_b(\zeta_{m_{n-1}}, \zeta_{m_n}), \forall \ n \geq 1
\]

So, the inequality (2) can be changed as 
\[
d_b(\zeta_{m_n}, \zeta_{m_{n+1}}) \leq \eta_s \left[ d_b(\zeta_{m_n}, \zeta_{m_{n-1}}) \right]^{\sigma} \cdot \left[ d_b(\zeta_{m_n}, \zeta_{m_{n+1}}) \right]^{\tau} \cdot \left[ d_b(\zeta_{m_{n-1}}, \zeta_{m_n}) \right]^{\rho} \\
\cdot \left[ d_b(\zeta_{m_{n-1}}, \zeta_{m_{n+1}}) \right]^{-\tau-\sigma-\rho} \\
\leq \eta_s \left[ d_b(\zeta_{m_{n-1}}, \zeta_{m_n}) \right]^{1-\tau}, \forall \ n \geq 1
\]

Now we shall reduce the equation as follows. Hence we derive it as, 
\[
d_b(\zeta_{m_n}, \zeta_{m_{n+1}}) \leq \eta_s [d_b(\zeta_{m_{n-1}}, \zeta_{m_n})] \\
\leq \eta_s^2 [d_b(\zeta_{m_{n-2}}, \zeta_{m_{n-1}})] \\
\leq \eta_s^3 [d_b(\zeta_{m_{n-3}}, \zeta_{m_{n-2}})] \\
\cdots \\
\leq \eta_s^n [d_b(\zeta_{m_{n}}, \zeta_{m_{n+1}})]
\]

From the assumption we derived as \( n < 1 \) by taking \( n \to \infty \) the inequality (4),

We shall derive that \( \varphi = 0 \),
It follows that we can prove that \( \{\zeta_{m_n}\} \) is a Cauchy sequence by deriving with the standard tools.

Starting with the triangle inequality we receive the following estimation, making use of the following standard results based on the inequality of complete b – metric space. Let \( q_b \geq 1 \) be the coefficient. Considering in the following equations we get,

\[
\begin{align*}
d_b(\zeta_{m_n}, \zeta_{m_n+r}) & \leq q_b \left[ (d_b(\zeta_{m_n}, \zeta_{m_n+1}) + d_b(\zeta_{m_n+1}, \zeta_{m_n+r})) \right] \\
& \leq q_b \left[ (d_b(\zeta_{m_n}, \zeta_{m_n+1}) + q_b \left[ d_b(\zeta_{m_n+1}, \zeta_{m_n+2}) \right] \right] \\
& \leq q_b \left[ d_b(\zeta_{m_n}, \zeta_{m_n+1}) + q_b \left[ d_b(\zeta_{m_n+1}, \zeta_{m_n+2}) \right] \right]
\end{align*}
\]

\[
\begin{align*}
& \leq q_b d_b(\zeta_{m_n}, \zeta_{m_n+1}) + q^2 b d_b(\zeta_{m_n+1}, \zeta_{m_n+2}) + q^3 b d_b(\zeta_{m_n+2}, \zeta_{m_n+3}) \\
& \quad + \ldots + q^{r-1} b d_b(\zeta_{m_n+r-1}, \zeta_{m_n+r}) \\
\leq q_b d_b(\zeta_{m_n}, \zeta_{m_n+1}) + q^2 b d_b(\zeta_{m_n+1}, \zeta_{m_n+2}) + q^3 b d_b(\zeta_{m_n+2}, \zeta_{m_n+3}) \\
& \quad + \ldots + q^{r-1} b d_b(\zeta_{m_n+r-1}, \zeta_{m_n+r}) \\
\leq q_b \eta_s d_b(\zeta_{m_n+1}, \zeta_{m_n}) + q^2 \eta_s d_b(\zeta_{m_n+2}, \zeta_{m_n}) + q^3 \eta_s d_b(\zeta_{m_n+3}, \zeta_{m_n}) \\
& \quad + \ldots + q^{r-1} \eta_s d_b(\zeta_{m_n+r}, \zeta_{m_n}) \\
\leq q_b \eta_s d_b(\zeta_{m_n+1}, \zeta_{m_n}) \left[ 1 + q_b \eta_s + q^2 \eta_s^2 + \ldots + q^{r-1} \eta_s^{r-1} \right] \\
\leq q_b \eta_s d_b(\zeta_{m_n+1}, \zeta_{m_n}) \left[ 1 + q_b \eta_s \right]^{-1}
\end{align*}
\]

(Since \( (1 - x)^{-1} = 1 + x + x^2 + \ldots \))

\[
\begin{align*}
d_b(\zeta_{m_n}, \zeta_{m_n+r}) & \leq \left( \frac{q_b \eta_s}{1 - q_b \eta_s} \right) d_b(\zeta_{m_n}, \zeta_{m_n+1}) \tag{7}
\end{align*}
\]

So, \( \{\zeta_{m_n}\} \) is a Cauchy sequence in the complete b-metric space \( (\varphi_b, d_b) \) and so, there exists \( \zeta_m \in \varphi_b \) such that \( \lim_{n \to \infty} d_b(\zeta_{m_n}, \zeta_m) = 0 \).

Supposing that \( \zeta_m \neq T_{b_m} \zeta_m \) for each \( n \geq 0 \), by letting \( \zeta_m = \zeta_{m_n} \) and \( \chi_m \) in (1).

Thus we get,

\[
\begin{align*}
d_b(\zeta_{m_n+1}, T_{b_m} \zeta_{m_n}) = d_b(T_{b_m} \zeta_m, T_{b_m} \zeta_m) \\
& \leq \eta_s \left[ d_b(\zeta_{m_n}, \zeta_m) \right]^\sigma \cdot \left[ d_b(\zeta_{m_n}, T_{b_m} \zeta_m) \right]^\tau \cdot \left[ d_b(\zeta_m, T_{b_m} \zeta_m) \right]^\rho \\
& \quad \cdot \left[ \left( d_b(\zeta_m, T_{b_m} \zeta_m) \right) + \left( d_b(\zeta_m, T_{b_m} \zeta_m) \right)^{\tau - \sigma} \right]^{-\rho} \tag{8}
\end{align*}
\]

Let \( n \to \infty \), in the inequality (7) we conclude that \( d_b(\zeta_m, T_{b_m} \zeta_m) = 0 \) which contradicts our assumption.

Thus, \( T_{b_m} \zeta_m = \zeta_m \)

Now let us prove its uniqueness.

If \( \zeta_m \) be any fixed point of \( T_{b_m} \) such that and apply it in equation (8).

We consider \( d_b(\zeta_m, \zeta_m) = d_b(T_{b_m} \zeta_m, T_{b_m} \zeta_m) \)
\[ \leq \eta_s [d_b(\zeta_m, \zeta_m')]^{\sigma'} \cdot [d_b(\zeta_m, T_{b_m} \zeta_m)]^{\tau'} \]
\[ [d_b(\zeta_m, T_{b_m} \zeta_m')]^{\sigma'} \cdot [(d_b(\zeta_m, T_{b_m} \zeta_m')] + [(d_b(\zeta_m, T_{b_m} \zeta_m)]^{\tau' - \sigma' - \rho'} \]
As \( n \to \infty \) in this inequality, \( d_b(\zeta_m, T_{b_m} \zeta_m) = 0 \).
It’s a contradiction so we get,
\[ T_{b_m} \zeta_m = \zeta_m \]
Therefore,
\[ \zeta_m = \zeta_m \]
Hence proved.

4. We can now derive the analog of the main theorem in the setting of Complete Partial b-Metric Spaces.
As we begin we shall have the lemma which is used in the following theorem.

**Lemma 4.1:**
Let \( p_b \) be a complete partial b-metric on a non-empty set \( \Phi_s \) and \( \varphi_{p_b} \) be the corresponding standard metric space on the same set \( \Phi_s \), then,
- A sequence \( \{\psi_{p_n}\} \) is the fundamental in the framework of a partial b-metric \( \langle \Phi_s, p_b \rangle \), if and only if it is a fundamental sequence in the setting of the corresponding standard metric space \( \langle \Phi_s, \varphi_{p_b} \rangle \).
- A partial b-metric space \( \langle \Phi_s, p_b \rangle \) is complete if and only if the corresponding standard metric space \( \langle \Phi_s, \varphi_{p_b} \rangle \) is complete.

Moreover,
\[ \lim_{n \to \infty} p_{p_b}(\psi_p, \psi_{p_n}) = 0 \iff p_b(\psi_p, \psi_p) = \lim_{n \to \infty} p_b(\psi_p, \psi_{p_n}) = \lim_{n,m \to \infty} p_b(\psi_{p_n}, \psi_{p_m}) \]
- If \( \psi_{p_n} \to \tau \) as \( n \to \infty \) in a partial b-metric space \( \langle \Phi_s, p_b \rangle \) with \( p_b(\tau, \tau) = 0 \), then we have,
\[ \lim_{n \to \infty} p_b(\psi_{p_n}, \rho_p) = p_b(\tau, \rho_p) \]
for every \( \rho_p \in \Phi_s' \).

So according to this Lemma, the sequence \( \{\psi_{p_n}\} \) is the fundamental sequence in the standard metric \( \langle \Phi_s, \varphi_{p_b} \rangle \).
Because, \( \langle \Phi_s, p_b \rangle \) is complete, \( \langle \Phi_s, \varphi_{p_b} \rangle \) is also complete.

**Theorem: 4.2**
Let \( \langle \Phi_s, p_b \rangle \) be a complete partial b-metric space with coefficient \( s_b \geq 1 \).
Let, \( T_{b_m}: \Phi_s \to \Phi_s \) be a given mapping. Suppose there exists \( \eta \in [0,1) \) and \( \tau, \sigma', \rho' \in (0,1) \) with \( \tau' + \sigma' + \rho' < 1 \), such that,
\[ p_b(T_{b_m} \varphi_p, T_{b_m} \varphi_p) \]
\[ \leq \eta \left[ \left(p_b(\psi_p, \varphi_p)\right)^{\sigma'} \cdot \left(p_b(\psi_p, T_{b_m} \varphi_p)\right)^{\tau'} \cdot \left(p_b(\varphi_p, T_{b_m} \varphi_p)\right)^{\rho'} \right] \]
\[ \cdot \left(1/2 \left[p_b(\psi_p, T_{b_m} \varphi_p) + p_b(\varphi_p, T_{b_m} \psi_p)\right] \right)^{1-\tau' - \sigma' - \rho'} \] (12)
For all $\Psi_p, \vartheta_p \in \phi_s \setminus Fix(T_{b_m})$.

**Proof:**

For any $\Psi_{p_0} \in (\Phi_s, p_b)$, we form a sequence $\{\Psi_{p_n}\}$ by $\Psi_{p_n} = T^{n b_m}(\Psi_{p_n})$ for each $n \in \mathbb{N}$.

If there exists $n_0$ such that $\Psi_{p_{n_0}} = \Psi_{p_{n_0+1}}$, then $\Psi_{p_{n_0}}$ is a fixed point of $T_{b_m}$.

So the proof gets completed.

So, now we shall assume that $\Psi_{p_n} = \Psi_{p_{n+1}}$ for each $n \geq 0$.

Through submitting the values of $\Psi_p = \Psi_{p_n}$ and $\vartheta_p = \Psi_{p_{n-1}}$ in (12),

So we get as ,

$$p_b(\Psi_{p_{n+1}}, \Psi_{p_n}) = p_b(T_{b_m} \Psi_{p_n}, T_{b_m} \Psi_{p_{n-1}})$$

$$\leq \eta \left[ p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) \right]^{\sigma} \cdot \left[ p_b(\Psi_{p_n}, T_{b_m} \Psi_{p_n}) \right]^{\tau} \cdot \left[ p_b(\Psi_{p_{n-1}}, T_{b_m} \Psi_{p_{n-1}}) \right]^{\rho}$$

$$\leq \eta \left[ p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) \right]^{\sigma} \cdot \left[ p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) \right]^{\tau} \cdot \left[ p_b(\Psi_{p_{n-1}}, \Psi_{p_{n-1}}) \right]^{\rho}$$

$$\leq \eta \left[ \frac{1}{2} \left( p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) + p_b(\Psi_{p_{n-1}}, \Psi_{p_{n+1}}) \right) \right]^{1-\tau-\sigma-\rho}$$

Since, $p_b(\Psi_{p_n}, \Psi_{p_{n-1}}) = 0$, we have,

$$\leq \eta \left[ \frac{1}{2} \left( p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) + p_b(\Psi_{p_{n-1}}, \Psi_{p_{n+1}}) \right) \right]^{1-\tau-\sigma-\rho}$$

$$\leq \eta \left[ \frac{1}{2} \left( p_b(\Psi_{p_n}, \Psi_{p_{n-1}}) + p_b(\Psi_{p_{n+1}}, \Psi_{p_{n}}) \right) \right]^{1-\tau-\sigma-\rho}$$

(13)

Supposing that,

$$[p_b(\Psi_{p_{n-1}}, \Psi_{p_n})] \leq [p_b(\Psi_{p_n}, \Psi_{p_{n+1}})]$$

So the inequality (13) becomes as,

$$[p_b(\Psi_{p_n}, \Psi_{p_{n+1}})] \leq \eta p_b(\Psi_{p_n}, \Psi_{p_{n+1}})$$

This is contradiction since we have considered that $\eta < 1$.

Hence we derive as the conclusion that,

$$[p_b(\Psi_{p_n}, \Psi_{p_{n+1}})] \leq [p_b(\Psi_{p_{n-1}}, \Psi_{p_n})], \quad \forall \ n \geq 1$$

Thus,

$$[p_b(\Psi_{p_{n-1}}, \Psi_{p_n})]$$

is a non – increasing sequence with positive terms.

The set we have,

$$\left[ \frac{1}{2} \left( p_b(\Psi_{p_{n-1}}, \Psi_{p_n}) + p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) \right) \right] \leq p_b(\Psi_{p_{n-1}}, \Psi_{p_n}), \forall \ n \geq 1$$

So, the inequality (12) can be changed as
\[ p_b(\psi_{p_n}, \psi_{p_{n+1}}) \leq \eta \left[ p_b(\psi_{p_{n-1}}, \psi_{p_n}) \right]^\sigma \cdot \left[ p_b(\psi_{p_{n+1}}, \psi_{p_n}) \right]^\tau \cdot \left[ p_b(\psi_{p_{n-1}}, \psi_{p_{n+1}}) \right]^{\rho} \]

\[ p_b(\psi_{p_n}, \psi_{p_{n+1}}) \leq \eta \left[ p_b(\psi_{p_{n-1}}, \psi_{p_n}) \right]^{1-\sigma}, \forall \ n \geq 1 \quad (14) \]

So, there exists a non-negative constant \( m \) such that,

\[ \lim_{n \to \infty} p_b(\psi_{p_{n-1}}, \psi_{p_n}) = m \] Here \( m \geq 0 \).

So, we get in the inequality (14) as,

\[ p_b(\psi_{p_n}, \psi_{p_{n+1}}) \leq \eta \gamma \left[ p_b(\psi_{p_{n-2}}, \psi_{p_{n-1}}) \right] \]

\[ \leq \eta \gamma \left[ p_b(\psi_{p_{n-3}}, \psi_{p_{n-2}}) \right] \]

\[ \leq \eta \gamma \left[ p_b(\psi_{p_{n-(n+1)}}, \psi_{p_{n-(n+1)-1}}) \right] \]

\[ \leq \eta \gamma \left[ p_b(\psi_{p_{0}}, \psi_{p_{1}}) \right] \quad (15) \]

Since \( \eta, \gamma < 1 \) we have \( \bar{\eta} = \eta \gamma < 1 \).

Thus letting \( n \to \infty \) in (15), we derive that \( m = 0 \).

It follows that, we can prove that \( \{ \psi_{p_n} \} \) is a fundamental Cauchy sequence by deriving with the standard tools.

Starting with the triangle inequality of the partial b-metric spaces, we receive the following estimation, making use of the following standard results based on the inequality of complete partial b-metric space.

Let \( q_b \geq 1 \) be the coefficient. Considering in the following equations we get,

\[ p_b(\psi_{p_n}, \psi_{p_{n+r}}) \leq s_b \left[ (p_b(\psi_{p_n}, \psi_{p_{n+1}}) + p_b(\psi_{p_{n+1}}, \psi_{p_{n+r}})) \right] \]

\[ \leq s_b \left[ (p_b(\psi_{p_n}, \psi_{p_{n+1}}) + p_b(\psi_{p_{n+1}}, \psi_{p_{n+2}}) + \ldots + s_b \left[ p_b(\psi_{p_{n+r}}, \psi_{p_{n+r}}) \right]) \right] \]

\[ \leq s_b p_b(\psi_{p_n}, \psi_{p_{n+1}}) + s_b^2 p_b(\psi_{p_{n+1}}, \psi_{p_{n+2}}) + \ldots + s_b^r p_b(\psi_{p_{n+r-1}}, \psi_{p_{n+r}}) \]

\[ \leq s_b p_b(\psi_{p_n}, \psi_{p_{n+1}}) + s_b^2 p_b(\psi_{p_{n+1}}, \psi_{p_{n+2}}) + \ldots + s_b^r p_b(\psi_{p_{n+r-1}}, \psi_{p_{n+r}}) \]
\[
\begin{align*}
&\leq s_b \tilde{\eta}^n p_b(\psi_{p_0}, \psi_{p_1}) + s_b^2 \tilde{\eta}^{n+1} p_b(\psi_{p_0}, \psi_{p_1}) \\
&+ s_b^3 \tilde{\eta}^{n+2} p_b(\psi_{p_0}, \psi_{p_1}) + \cdots + s_b^r \tilde{\eta}^{n+r-1} p_b(\psi_{p_0}, \psi_{p_1}) \\
&\leq s_b \tilde{\eta}^n p_b(\psi_{p_0}, \psi_{p_1}) \{1 + s_b \tilde{\eta} + s_b^2 \tilde{\eta}^2 + \cdots + s_b^{r-1} \tilde{\eta}^{r-1}\}^{-1} \\
&\leq s_b \tilde{\eta}^n p_b(\psi_{p_0}, \psi_{p_1}) \left(\frac{1}{1 - s_b \tilde{\eta}}\right)
\end{align*}
\]

(Since \((1 - x)^{-1} = 1 + x + x^2 \ldots \))

\[
p_b(\psi_{p_n}, \psi_{p_{n+r}}) \leq \left(\frac{s_b \tilde{\eta}^n}{1 - s_b \tilde{\eta}}\right) p_b(\psi_{p_0}, \psi_{p_1})
\]

(17)

The sequence \(\{\psi_{p_n}\}\) is the fundamental sequence in \((\Phi_s, p_b)\), as \(n \to \infty\)

So according to the Lemma, the sequence \(\{\psi_{p_n}\}\) is the fundamental sequence in the standard metric \((\Phi_s, q_{p_b})\).

Because, \((\Phi_s, p_b)\) is complete, \((\Phi_s, q_{p_b})\) is also complete.

So there exists, \(\psi_p \in \Phi_s\) such that,

\[
p_b(\psi_p, \psi_p) = \lim_{n \to \infty} p_b(\psi_p, \psi_{p_n}) = \lim_{n \to \infty} p_b(\psi_{p_n}, \psi_{p_m}) = 0
\]

(18)

Thus we get,

\[
\lim_{n \to \infty} q_{p_b}(\psi_p, \psi_{p_n}) = 0
\]

(19)

Now we shall show that the limit \(\psi_p\) of the iterative sequence \(\{\psi_{p_n}\}\) is a fixed point of the mapping \(T_{b_m}\).

Taking it as the assumption is that we get,

\[
\psi_p \neq T_{b_m} \psi_p
\]

(20)

So that we derive it as,

\[
p_b(\psi_p, T_{b_m} \psi_p) > 0
\]

We know that, \(\psi_{p_n} \neq T_{b_m} \psi_p\) for each \(n \geq 0\), as we let \(\psi_p = \psi_{p_n}\) in (12) we get that,

\[
p_b(\psi_{p+1}, T_{b_m} \psi_p) = p_b(T_{b_m} \psi_{p_n}, T_{b_m} \psi_p)
\]

\[
\leq s_b \tilde{\eta}^n p_b(\psi_{p_n}, \psi_{p_1}) \left[\frac{1}{2} [p_b(\psi_{p_n}, T_{b_m} \psi_{p_n}) + p_b(\psi_p, T_{b_m} \psi_{p_n})]\right]^{1-\alpha-\beta} \\
\]

\[
= s_b \tilde{\eta}^n p_b(\psi_{p_n}, \psi_{p_1}) \left[\frac{1}{2} [p_b(\psi_{p_n}, T_{b_m} \psi_{p_n}) + p_b(\psi_p, T_{b_m} \psi_{p_n})]\right]^{1-\alpha-\beta}
\]

As \(n \to \infty\) in the inequality (21) we can find that ,

\[
p_b(\psi_p, T_{b_m} \psi_p) = 0
\]

So, \(\psi_p = T_{b_m} \psi_p\)
Which contradicts our assumption, so, we find that,

\[ T_{b_m} \Psi_p = \Psi_p. \]

Hence the theorem.

**References:**