COMMON FIXED POINT THEOREMS ON LOCALLY CONTRACTIVE MAPPINGS IN b-MULTIPLICATIVE METRIC SPACES

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Abstract

This paper deals with the mappings having the unique common fixed point in the closed ball in b-multiplicative metric spaces and also common fixed points of locally contractive mappings in b-multiplicative metric space with applications.

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1. INTRODUCTION

A metric space is a non-void set together with the structure established by a well-defined notation of distance. The word 'metric'is introduced from the word **metor** (measure). The distance functions are learned as metric in mathematical literature.

In 1905, the distance functionswere first introduced by Maurice Frechet (French mathematician) and he interested in distance of generalizing concepts and advancing them to arbitrary sets.The source of fixed point theory belonging to the approach of successive Approximation meant for proving the presence of solutions of Differential equation introduced by Joseph Liouville in 1873 and Charles Emile picard in 1890.But it was perfectly initiated in the twentieth century (beginning) as an influencial part of analysis. The fixed point theory is a beautiful mixture of analysis (pure and applied), topology and geometry.

The fixed point and common fixed point theorems for difference types of non-linearContractive functions have been determined by numerous researchers.

Fixed point problems containing various kinds of inequalities like that Schwarz inequality,minkowski's inequality,Cauchy inequality and it is obtained by various author.

In recent times,various researchers have determined the fixed point, paired fixed point,Common fixed point and the theorems on four self mapping having the unique fixed point in b-multiplicative metric space and the other spaces.

The idea of weakly commutative mapping and the proof of some common Fixed point theorems for these mappings was proposed by Jungck and Rhoades.

The multiplicative metric space was established by Bashirov et al., during the period of 2008. The two person named Ozavsar and Cervikel determined the notion of convergence in multiplicative metric space (MMS) and they both studied some fixed point results in that space.

2. PRELIMINARIES

Definition 2.1: (Multiplicative metric space)[1]

Let Ω be a non-empty set. Let $\Delta:\Omega^2 \to R^+$ be a function is said to be **multiplicative metric** on Ω provided for any s, t, uin Ω the below conditions are satisfied:

- $\Delta(s,t) \geq 1$ and $\Delta(s,t)=1$ iff s= t
- $\Delta(s,t)=\Delta(t, s)$
- $\Delta(s,t) \leq \Delta(s, u).\Delta(u, t)$

Then the pair (Ω,Δ)is known as **multiplicative metric space.**

Example 2.2:

Let $\mathbb{R}_p^+$ be the collections of all p tuples of positive real numbers and Ω equal to $\mathbb{R}_p^+$. Then $\Delta(s,t)=\left|s_1/t_1, s_2/t_2, \dots\dots\dots\dots\dots\dots s_p/t_p\right|$ it gives a **multiplicative metric** on Ω.

Definition 2.3: (b-multiplicative metric space)

Let P be a nonempty set and q ≥1 be a given real number. A mapping $r:\Omega \times \Omega \to [0,\infty)$ is said to be **b-multiplicative metric** if the following condition satisfied:

a) r (s, t) >1 for all s, t ∈ P with s ≠ t and r (s, t)=1 if and only if s = t
b) r(s,t) = r(t,s) for all s,t ∈ P
c) $r(s,u) \leq r(s,t)^q \,.\, r(t,u)^q$

The triplet (P,r,q) is said to be a **b-multiplicative metric space**.

Example 2.4:

Let Ω=[0,∞). Let $\mathcal{H}_h$ be a mapping $\mathcal{H}_h$:Ω x Ω → [1,∞) by $\mathcal{H}_h(s,t)=h^{(s-t)^2}$, Here fixed h >1 be a real number. Then for each a in Ω.$\mathcal{H}_h$ is a b-multiplicative metric space with q=2.Here we see that$\mathcal{H}_h$is a not a multiplicative metric space.

Definition 2.5:**(Multiplicative convergent)**

Let Ω be a multiplicative metric space and choose an arbitrary point S_0in Ω and μ>1. Let B(S_0, μ) be themultiplicative open ball with radiusμ centered atS_0 is the set

{t∈ Ω : Δ(t,S_0)<μ}.

Let $\{S_n\}$ be a sequence in multiplicative metric space Ω is said to be **multiplicative convergent** to a point s in Ω provided for any given μ>1, there is n in N such that $S_n$in B(s, μ) for all n in N. Let $\{S_n\}$ be a sequence in Ω is said to be multiplicative convergent to s in Ω if and only if $\Delta(S_n, s) \to 1$ as n→ ∞.

Definition 2.6:(Multiplicative continuous)[3]

Let (Ω, Δ_Ω) and (λ, Δ_λ) be the multiplicative metric spacesand choose an arbitrary fixed point s_0 in Ω.Define a mapping h: Ω → λ is called a **multiplicative continuous** at $s_0$iff$s_n \to s_0$ in (Ω,Δ_Ω) implies h(s_n)→h(s_0) in (λ,Δ_λ) for every {s_n } in Ω. Let μ>1be given, there exists$\rho(s_0, \mu)$>1 such that Δ_λ(hs,hs_0) <μ whenever Δ_Ω(s,s_o)<ρ for s in Ω.

Definition 2.7:(Multiplicative Cauchy sequence and complete)[4]

Let (Ω, Δ) be a multiplicative metric space.

Let $\{s_n\}$ be a sequence in Ω is called **multiplicative Cauchy sequence** if for any μ>1, there exists n in N such that Δ (s_i,s_j) $\leq \mu$ for all i, j > N.

Let (Ω, Δ) be multiplicative metric space and it is said to be **complete** provided that every Cauchy sequence $\{s_n\}$ in Ω is multiplicative convergent to a point s in Ω.

Definition 2.8:(Multiplicative Cauchy)[3]

Let (Ω, Δ) be a multiplicative metric space.Let a sequence $\{s_i\}$ in Ω is said to be **multiplicative Cauchy**iff $\Delta(s_i,s_j) \to 1$as i,j →∞.

Definition 2.9: (Fixed point)[1]

Let α, β: Ω→ Ω be the maps. Let choose a point s in Ω is said to be

- **Fixed point** of h if hs = s;
- Coincidence point of the pair {α,β} if αs=βs;
- The pair{α, β} have the **common point** if s = αs=βs.

Definition 2.10:(β-multiplicative contraction)[6]

Let (Ω,Δ) be a complete multiplicative metric space and α,β:Ω→ Ω. The mapping α is said to be **β-multiplicative contraction** if there exists m in [0, 1) such that $\Delta(\alpha s,\alpha t)\leq(\Delta(\beta s,\beta t))^m$for all s,t in Ω.

Definition 2.11:(Multiplicative contractive)[2]

Let (Ω,Δ) be a complete multiplicative metric space. A mapping α:Ω→Ω is called a **multiplicative contractive** if there existsδ in [0,1) such that

$\Delta(\alpha s,\alpha t)\leq(\Delta(s,t))^\delta$for all s,t in Ω.

Definition 2.12:(Weakly commutative)[5]

Let (Ω,Δ) be a complete multiplicative metric space. Consider a mapping α, β:Ω→ Ω. The pair (α,β) is said to be

Commutative if α β s=β α s,**weakly commutative** if Δ (α β s, β α s) ≤ Δ (α s, β s) for all s in Ω.

3.MAIN RESULTS

In this section we discuss some results on "Common fixed points of locally contractive mappings in b-multiplicative metric spaces " and these can be extended to results on"Common fixed points for weak commutative mappings on a multiplicative metric spaces ".

THEOREM 3.1:

Let Ω be the complete b- multiplicative metric space and define α and β are two maps on a complete b-multiplicative metric space Ω.In Ω choose s_0 an arbitrary point. Assume that there exists μ in [0,1) such that

$$\Delta(\alpha s,\beta t) \leq \Delta(s,t)^{\mu} \quad (3.1)$$

for any s,t in $\overline{B_g(s_0, a)}$ and $\Delta(s_0, \alpha(s_0)) \leq a^{(1-\mu)}$ is satisfied.In $\overline{B_g(s_0, a)}$,there exist a unique common fixed point of α and β.

Proof

Let s_0 be a given point in Ω.

Consider a sequence $\{s_i\}$ in Ω such that $s_{2i+1}=\alpha(s_{2i})$ and $s_{2i+2}=\beta(s_{2i+1})$ for all i≥0.

We assert that s_i in $\overline{B_g(s_0, a)}$ for all i in N. Note that

$\Delta(s_0,s_1)= \Delta(s_0,\alpha(s_0)) \leq a^{(1-\mu)} \leq$ a. (3.2)

$\Rightarrow s_1$ in $\overline{B_g(s_0, a)}$. Let $s_2, s_3 \ldots\ldots\ldots\ldots s_n$ in $\overline{B_g(s_0, a)}$

for some n in N. Clearly, if n=2m+1,then

$$\begin{aligned}\Delta(s_{2m+1},s_{2m+2}) &= \Delta(\alpha s_{2m},\beta s_{2m+1})\\ &\leq \Delta(s_{2m},s_{2m+1})^{\mu}\\ &\leq \Delta(s_{2m-1},s_{2m})^{\mu^2}\\ &\ \ .\\ &\ \ .\\ &\ \ .\\ &\leq \Delta(s_0,s_1)^{\mu^{2m+1}} \quad (3.3)\end{aligned}$$

In this same way, if n=2m+2, then

$$\begin{aligned}\Delta(s_{2m+2},s_{2m+3}) &= \Delta(\alpha s_{2m}, \beta s_{2m+2})\\ &\leq \Delta(s_{2m},s_{2m+2})^{\mu}\\ &\leq \Delta(s_{2m-2},s_{2m})^{\mu^2}\\ &\leq \Delta(s_0,s_1)^{\mu^{2m+2}} \quad (3.4)\end{aligned}$$

Hence for any m in N, we obtain

$$\Delta(s_m,s_{m+1}) \leq \Delta(s_0,s_1)^{\mu^m}$$

Now, using triangular inequality for b-multiplicative metric space and q≥1.We have

$$\Delta(s_0,s_{m+1}) \leq \Delta(s_0,s_1)^q.\Delta(s_1,s_2)^q.\Delta(s_2,s_3)^q\Delta \ldots\ldots\ldots(s_m,s_{m+1})^q \quad (3.4)$$

whichimplies that

$$\Delta(s_0,s_{m+1}) \leq \Delta(s_0,s_1)^q.\Delta(s_0,s_1)^{q\mu}.\Delta(s_0,s_1)^{q\mu^2}\ldots\ldots\Delta(s_0,s_1)^{q\mu^m}$$

Where q≥1,

Thus we get the term,

$$\begin{aligned}\Delta(s_0,s_{m+1}) &\leq \Delta(s_0,s_1)^q.\Delta(s_0,s_1)^{q\mu}.\Delta(s_0,s_1)^{q\mu^2}\ldots\ldots\ldots\ldots\ \Delta(s_0,s_1)^{q\mu^m}\\ &\leq \Delta(s_0,s_1)^{(q+q\mu+q\mu^2\ldots\ldots\ldots q\mu^m)}\\ &\leq \Delta(s_0,s_1)^{q(1+\mu+\mu^2\ldots\ldots\ldots\mu^m)}\\ &\leq \Delta(s_0,s_1)^{q(\frac{1-\mu^{m+1})}{(1-\mu)}}\end{aligned}$$

Where q≥1, we know that if s_1 in $\overline{B_g(s_0, a)}$

Then $\Delta(s_0,s_{m+1}) \leq (a^{(1-\mu)q(\frac{1-\mu^{m+1})}{(1-\mu)}}) \leq a^{(1-\mu^{m+1})} \leq a$ for all m in N.

Hence s_{m+1} in $\overline{B_g(s_0, a)}$.By induction on i,we deduce that $\{s_i\}$ in $\overline{B_g(s_0, a)}$ for all i in N. Now we prove that $\{s_i\}$ is Cauchy in $\overline{B_g(t_0, a)}$.Therefore for each c, d in N such that k>i, Now, using triangular inequality for b-multiplicative metric space, and q≥1

$\Delta\ (s_i,s_k) \leq \Delta\ (s_i,s_{i+1})^q.\Delta\ (s_{i+1},s_{i+2})^q.\ \ldots\ldots\ldots\ldots\ldots\ \Delta\ (s_{k-1},s_k)^q$

$\leq \Delta\ (s_0,s_1)^{q\mu^i}.\Delta\ (s_0,s_1)^{q\mu^{i+1}}.\Delta\ (s_0,s_1)^{q\mu^{i+2}}\ldots\ldots\ \Delta\ (s_0,s_1)^{q\mu^{k-1}}$

$\leq \Delta\ (s_0,s_1)^{q(\mu^i+\mu^{i+1}+\cdots+\mu^{k-1})}$

$\leq \Delta\ (s_0,s_1)^{q(\mu^i+\mu^{i+1}+\cdots)}$

$\leq \Delta\ (s_0,s_1)^{q(\frac{(\mu^i)}{(1-\mu)}}$

where q≥1, taking limit as k,i →∞ we obtain Δ (s_k,s_i) →1. Hence $\{s_i\}$ is a multiplicative Cauchy sequence. By the completeness of Ω (we know that every Cauchy sequence is convergent) it follows that $\lim_{i\to\infty} s_i$= p for a point pin$\overline{B_g(s_0,\mathrm{a})}$.

Also,we have

$\Delta\ (s_{2i+1},\ \beta\mathrm{p}) =\ \Delta\ (\alpha s_{2i},\ \beta\mathrm{p}) \leq \Delta\ (s_{2i},\mathrm{p})^{\mu}$(3.6)

Taking limit as i →∞ on the both sides of the equation (6)

$\Delta\ (\mathrm{p},\ \beta\mathrm{p}) \leq \Delta\ (\mathrm{p},\mathrm{p})^{\mu}$

Thus, p is a fixed point of β. In this same manner,we can note that p is the fixed point of α. Hence we conclude that p is common fixed points of α and β in$\overline{B_g(s_0,\mathrm{a})}$. Indeed, if v is another fixed point of α and β, then

$\Delta(\mathrm{p},\mathrm{v}) = \Delta\ (\alpha\mathrm{p},\ \beta\mathrm{v}) \leq \Delta\ (\mathrm{p},\mathrm{v})^{\mu}$which implies that p=v.

Hence proved.

THEOREM 3.1.2:

Let Ω be the complete b-multiplicative metric space and define ξ,ψ,α and β are the self-maps of a complete b- multiplicative metric space Ω and (α,ξ) and (β,ψ) are weakly commutative with ξΩ⊂ βΩ, ψΩ⊂ αΩ, and one of ξ, ψ, α and β is continuous. In Ω for some given point s_0 if $\xi s_0=t_0$and there exists δ in $(0,\frac{1}{2})$ with $\mathrm{v}=\frac{\delta}{1-q\delta}$ such that

$\Delta(\xi s,\psi s) \leq (\mathcal{H}(s,t))^{\delta}$ for any s,t ∈ $\overline{B_g(t_0,\mathrm{a})},$ (3.7)

Is satisfied

where $\mathcal{H}$(s,t)=max{Δ(αs,βt),Δ(αs,ξs),Δ(βt,ψt),Δ(ξs,βt),Δ(αs,ψt)}. In $\overline{B_g(t_0,\mathrm{a})},$ there exists a unique common fixed point of α,ψ,ξ and β it provided that $\Delta(t_0,\psi s_1) \leq a^{(1-v)}$ for some s_1 in Ω.

Proof

Let given point s_0 in Ω. Given ξΩ ⊂βΩ, let us choose a point s_1 in Ω such that $\xi s_0=\beta s_1=t_0$. Similarly, there exists a point s_2 in Ω such that $\psi s_1=\alpha s_2=t_1$. Indeed it follows from the assumption that ψΩ⊂ αΩ. Let us consider the sequences $\{s_i\}$ and $\{t_i\}$ in Δ such that

$t_{2i}=\xi s_{2i}=\beta s_{2i+1}$, $t_{2i+1}=\psi s_{2i+1}=\alpha s_{2i+2}$

Now we assert that $\{t_i\}$ is a sequence in $\overline{B_g(t_0,\mathrm{a})}$,Note that

$\Delta(t_0,t_1) = \Delta(t_0,\psi s_1) \leq a^{(1-v)} <$ a. Hencet_1 in $\overline{B_g(t_0,\mathrm{a})},$ we assume that $s_2,\ldots\ldots\ldots\ldots\ldots.s_n$ in $\overline{B_g(t_0,\mathrm{a})},$ for some n in ℕ. Then, if n=2m, it follows that from equation (3.7)

$\Delta(t_{2m},t_{2m+1}) = \Delta(\xi s_{2m},\psi s_{2m+1})$

$\leq$ (max ({$\Delta(\alpha s_{2m},\beta s_{2m+1}),$

$\Delta(\alpha s_{2m},\xi s_{2m}),\Delta(\beta s_{2m+1},\psi s_{2m+1}),\Delta(\xi s_{2m},\beta s_{2m+1}),\Delta(\alpha s_{2m},\psi s_{2m+1})\})^{\delta}$ $\leq$(max $(\{\Delta(t_{2m-1},t_{2m}),$

$\Delta(t_{2m-1},t_{2m}),\Delta(t_{2m},t_{2m+1}),\Delta(t_{2m},t_{2m}),\Delta(t_{2m-1},t_{2m+1})\})^{\delta}$

$\leq(\max(\{\Delta(t_{2m-1},t_{2m}), \Delta(t_{2m},t_{2m+1}), 1, \Delta(t_{2m-1},t_{2m+1})\})^{\delta}$

Now, using triangular inequality for b-multiplicative metric space and q≥1

We have

$\leq (\max(\{\Delta(t_{2m-1},t_{2m}),$

$\Delta(t_{2m}, t_{2m+1}),1,\Delta(t_{2m-1}, t_{2m})^{q}.\Delta(t_{2m}, t_{2m+1})^{q}\})^{\delta}$

$= \Delta(t_{2m-1},t_{2m})^{q\delta}.\Delta(t_{2m}, t_{2m+1})^{q\delta}$ Where q≥1,

Implies that $\Delta(t_{2m}, t_{2m+1}) \leq (\Delta(t_{2m-1}, t_{2m}))^{v}$ for all m in N, where $v=\frac{\delta}{1-q\delta}$

In this same way assume, n=2m+1 and we get

$\Delta(t_{2m+1}, t_{2m+2}) \leq (\Delta(t_{2m},t_{2m+1})^{v}$

Hence

$\Delta(t_{m}, t_{m+1}) \leq \Delta(t_{m-1}, t_{m})^{v}$ for m in N

Therefore

$\Delta(t_{m}, t_{m+1}) \leq \Delta(t_{m-1},t_{m})^{v} \leq \Delta((t_{m-2},t_{m-1})^{v^2} \leq \ldots\ldots\ldots \leq (t_0,t_1)^{v^m}$

for all m in N.

Now, using triangular inequality for b-multiplicative metric space and q≥1. We have

$\Delta(t_0, t_{m+1}) \leq \Delta(t_0,t_1)^{q}.\Delta(t_1,t_2)^{q}.\Delta(t_2,t_3)^{q} \ldots\ldots\ldots \Delta(t_m,t_{m+1})^{q}$

Thus

$\Delta(t_0, t_{m+1}) \leq \Delta(t_0,t_1)^{q}.\Delta(t_0,t_1)^{qv}.\Delta(t_0,t_1)^{qv^2} \ldots\ldots\ldots \Delta(t_0,t_1)^{qv^m}$

$\leq \Delta(t_0,t_1)^{q(1+v+\cdots v^m)}$

$\leq \Delta(t_0,t_1)^{q\frac{(1-v^{m+1})}{(1-v)}}$

where q≥1, since t_1 in $\overline{B_g(t_0, a)}$, we have the inequality is

$\Delta(t_0, t_{m+1}) \leq (a^{(1-v)})^{q\frac{(1-v^{m+1})}{(1-v)}} \leq a^{(1-v^{m+1}}) \leq$ a, for all m in N this implies that $t_{m+1} \in \overline{B_g(t_0, a)}$, by induction on i, we conclude that $\{t_i\}$ in $\leq \overline{B_g(t_0, a)}$, for all i in N.

Now we show that the sequence $\{t_i\}$ satisfies the multiplicative Cauchy criterion for convergence $(\overline{B_g(t_0, a)},\Delta)$. Let i,j in N such that j>i,now using triangular inequality for b-multiplicative metric space and q ≥ 1 we have

$\Delta(t_i, t_j) \leq \Delta(t_i,t_{i+1})^{q}.\Delta(t_{i+1},t_{i+2})^{q}.\Delta(t_{i+2},t_{i+3})^{q} \ldots\ldots\ldots \Delta(t_{j-1},t_j)^{q}$

$\leq \Delta(t_0,t_1)^{qv^i}.\Delta(t_0,t_1)^{qv^{i+1}}.\Delta(t_0,t_1)^{qv^{i+2}} \ldots\ldots\ldots \Delta(t_0,t_1)^{qv^{i-1}}$

$\leq \Delta(t_0,t_1)^{q(v^i+v^{i+1+\cdots})}$

$\leq \Delta(t_0,t_1)^{q(\frac{v^i}{(1-v)})}$ Where q≥1,

Consequently, $\Delta(t_i, t_j) \to 1$ as i,j→∞. Hence the sequence $\{t_i\}$ is a multiplicative Cauchy sequence.

Given Δ is complete so is $\overline{B_g(t_0, a)}$. Hence $\{t_i\}$ has a limit point, taken u is a limit point in $\overline{B_g(t_0, a)}$ and must $\{t_i\}$ has a subsequence.

Let $\{\xi s_{2i}\} = \{\beta s_{2i+1}\} = \{t_{2i}\}$ and $\{\psi s_{2i+1}\} = \{\alpha s_{2i+2}\} = \{t_{2i+1}\}$ are the Subsequences of $\{t_i\}$ makes

$\lim_{i\to\infty} \xi s_{2i}$ =$\lim_{i\to\infty} \beta s_{2i+1}$= $\lim_{i\to\infty} \psi s_{2i+1}$=$\lim_{i\to\infty} \alpha s_{2i+2}$ = p.

Let us assume that α is continuous; then

$\lim_{i\to\infty} \alpha(\xi s_{2i})$=$\alpha(\lim_{i\to\infty} \xi s_{2i})$= $\alpha(\lim_{i\to\infty} \xi s_{2i+2})$= α (p).

The pair {α, ξ} are weak commutative, we get

$\Delta(\alpha(\xi s_{2i}), \xi(\alpha s_{2i})) \leq \Delta(\alpha s_{2i}, \xi s_{2i})$ (3.8)

Taking limit as i→∞ on the both sides of the above inequality,we get

Δ(α(p),$\lim_{i\to\infty} \xi(\alpha s_{2i})$)≤ Δ(p,p),

Which implies that $\lim_{i\to\infty} \xi(\alpha s_{2i})$=α(p). Using the condition (3.7),we have

$\Delta(\xi(\alpha s_{2i}),\psi s_{2i+1}) \leq (\max(\{\Delta(\alpha^2 s_{2i},\beta s_{2i+1}),$

$\Delta(\alpha^2 s_{2i},\xi s_{2i}),\Delta(\beta s_{2i+1},\psi s_{2i+1}),\Delta(\xi s_{2i},\beta s_{2i+1}),\Delta(\alpha^2 s_{2i},\psi s_{2i+1})\})^{\delta}$(3.9)

Taking limit as i→ ∞ on the both sides of (3.8), we have

$\Delta(\alpha p,p) \leq (\max\{\Delta(\alpha p,p),\Delta(\alpha p,\alpha p),\Delta(p,p), \Delta(\alpha p,p), \Delta(\alpha p,p)\})^{\delta}$

Which implies that Δ(αp,p)≤ $\Delta(\alpha p,p)^{\delta}$ we conclude that Δ(αp, p)=1 and p is a fixed point of α. In this same way using the condition (3.7) we obtain

$\Delta(\xi(p),\psi s_{2i+1}) \leq (\max(\{\Delta(\alpha p,\beta s_{2i+1}),$

$\Delta(\alpha p,\xi p),\Delta(\beta s_{2i+1},\psi s_{2i+1}),\Delta(\xi p,\beta s_{2i+1}),\Delta(\alpha p,\psi s_{2i+1})\})^{\delta}$(3.10)

Taking limit as i →∞ we obtain,

$\Delta(\xi p,p) \leq (\max(\{\Delta(\alpha p,p), \Delta(p,\xi p),\Delta(p,p),\Delta(\xi p,p),\Delta(p,p)\})^{\delta}$

Δ(ξp,p)≤ $\Delta(\xi p,p)^{\delta}$hence Δ(ξp,p)=1 and p is the fixed point of β.

Because of the fact that p=ξ(p) ∈ $\xi(\overline{B_g(t_0, a)}) \subseteq \beta(\overline{B_g(t_0, a)})$,

letp^* in $\overline{B_g(t_0, a)}$be suchthat p=β(p^*). So it follows from the equation (3.7)

$\Delta(p,\psi p^*)=\Delta(\xi(p),\psi p^*)$

$\leq (\max(\{\Delta(\alpha p,\beta p^*),$

$\Delta(\alpha p,\xi p),\Delta(\beta p^*,\psi p^*),\Delta(\xi p,\beta p^*),\Delta(\alpha p,\psi p^*)\})^{\delta}$,

Hence Δ(p,ψp^*)=1 and p=ψp^*. We know that {α,ψ} is weakly commutative from our assumption,thus

$\Delta(\beta p,\psi p)= \Delta(\beta\psi p^*,\psi\beta p^*) \leq \Delta(\beta p^*,\psi p^*)=\Delta(p,p)=1$.

Hence βp=ψp again using the eqn (*),we get

$\Delta(p,\psi p^*)=\Delta(\xi(p),\psi p)$

$\leq(\max(\{\Delta(\alpha p,\beta p), \Delta(\alpha p,\xi p),\Delta(\beta p,\psi p),\Delta(\xi p,\beta p),\Delta(\alpha p,\psi p)\})^{\delta}$ (3.11)

Hence p=ψp^* and u is common fixed point α,β,ξ and ψ in $\overline{B_g(t_0, a)}$.

If β is continuous,then we can proceed in this same arguments,we obtain that p=α(p)=β(p)=ψ(p)=ξ(p). Now assume that ξ is continuous.Thus

$\lim_{n\to\infty} \xi(\alpha s_{2i})$=ξ($\lim_{i\to\infty} \xi s_{2i}$)=ξ(p).

Since the pair {α, ξ} is weakly commuting, we obtain

$\Delta(\alpha(\xi s_{2i}),\xi(\alpha s_{2i}))\leq\Delta(\alpha s_{2i},\ \xi s_{2i})$(3.12)

Taking limit as i→∞ on the above equation we obtain

$\Delta\ (\lim_{i\to\infty}\alpha(\xi s_{2i}),\xi p)\ \leq\Delta(p,p)=1$ and $\lim_{i\to\infty}\alpha(\xi s_{2i})=\xi(p)$.

By condition (3.7) we have

$\Delta(\xi(\xi s_{2i}),\psi s_{2i+1})\leq(\max(\{\Delta(\alpha\,\xi s_{2i},\beta s_{2i+1}),$

$\Delta(\alpha\,\xi s_{2i},\alpha\,\xi s_{2i}),\Delta(\beta s_{2i+1},\psi s_{2i+1}),\Delta(\xi\xi s_{2i},\beta s_{2i+1}),\Delta(\alpha\,\xi s_{2i},\psi s_{2i+1})\})^{\delta}$ (3.13)

Taking limit as i→∞ on the both sides of (3.13)we obtain $\Delta(\xi p,p)\leq\Delta(\xi p,p)^{\delta}$

Hence Δ(ξp,p)=1 and p is a fixed point of ξ in $\overline{B_g(t_0,a)}$. We know that that $p=\xi(p)\in\xi\ (\overline{B_g(t_0,a)})\subseteq\beta(\ \overline{B_g(t_0,a)})$,let p^* in $\overline{B_g(t_0,a)}$ be such that $p=\beta(p^*)$. Again using equation (3.7)

$\Delta(\xi(\xi s_{2i}),\psi p^*)$

$\leq(\max(\{\Delta(\alpha\,\xi s_{2i},\beta p^*),$

$\Delta(\alpha\,\xi s_{2i},\alpha\,\xi s_{2i}),\Delta(\beta p^*,\psi s_{2i+1}),\Delta(\xi\xi s_{2i},\beta p^*),\Delta(\alpha\,\xi s_{2i},\psi p^*)\})^{\delta}$ (3.14)

Taking limit as i→∞ on the both sides of (3.14) implies that

$\Delta(p,\psi p^*)\leq\Delta(p,\psi p^*)^{\delta}$. Hence $p=\psi p^*$. Since the pair {ψ,β} is weakly commutative then,

Δ(ψp,β□)= Δ(ψβ□*,□□□*)≤ Δ(ψ□*,β□*)=Δ(p,p)=1.

Which implies that ψp=β □. from (3.7)we get

Δ((ξ□*2□*),ψ□*)

≤(max({Δ(α□*2□*,β□*),Δ(α□*2□*,α□*2□*),Δ(β□*,ψ□*2□+1*),Δ(ξ□*2□*,β□*), Δ(α(α□*2□*,ψ□*)})□

(3.15)

Taking limit as i→∞ on the both sides of (3.15) implies that

Δ(p,ψp)≤ Δ(p,ψ□)□

Hence p = ψp.

p=ψ(p) ∈ □ (□□(□*0*, a))⊆α(□□(□*0*, a)),let r in □□(□*0*, a) be such that p=α(r).

Again using equation (3.7) the we obtain the below equation

Δ((ξr),p)

≤(max ({Δ(αr,βp),Δ(ξr,βp), Δ(βp,ψp),Δ(ξr,βp),Δ(αr,ψp)})□ (3.16)

Implies that Δ((ξr),p)≤ Δ((ξr),p)□

Hence ξr=p.

We know that {ξ,ψ} is weakly commutative from our assumption,thus

Δ(ξp,ψ□)=Δ(βψr,ψ□□)≤Δ(βr,ψ□)=Δ(p,p)=1and ξp=ψ□.

Applying condition (3.7),we obtain

Δ(ξp,p)=Δ(ξp,ψp)

≤(max({Δ(αp,βp), Δ(αp,ξp),Δ(βp,ψp),Δ(ξp,βp),Δ(αp,ψp)})□(3.17)

= (max ({Δ(ξp,p), Δ(ξp,ξp),Δ(βp,ψp),Δ(ξp,p),Δ(ξp,ψp)})□

Hence ξ p=p and p is a common fixed point of α,β,ψ and ξ in(□□(□*0*, a)).

To assert that the uniqueness of the common fixed point of the maps α,β, ψ and ξ so, choose y in $\overline{\square_{\square}(\square_0, a)}$is the another common fixed point of α,β,ψ and ξ.

Δ(p,y)=Δ(ξp,ψy)

≤(max ({Δ(αp,βy), Δ(αp,ξp),Δ(βy,ψy),Δ(ξp,βp),Δ(αp,ψy)}) $^{\square}$

That isΔ(p,y)≤ Δ(p,y)$^{\square}$.We conclude that p=y and this implies that the common fixed points of α,β,ψ and ξ is unique.

Hence proved.

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