On Hyperbolic Hsu-structure Metric Manifold

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Abstract: In this paper we have proved theorems of different kinds in n-recurrent and n-recurrent symmetric hyperbolic Hsu-structure metric manifolds involving equivalent conditions with respect to associated Pseudo H-Projective curvature tensor, associated Pseudo Bochner curvature tensor and Ricci tensor.

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1. Introduction

Let us consider a differentiable manifold

Consider a differentiable manifold $M_n$ of differentiability class $C^\infty$ [1]. Let there exist in $M_n$ a vector valued linear function $F$ of class $C^\infty$ [2], satisfying the algebraic equation [8]:

\[(1.1a) \quad \bar{X} = a'X, \text{ for arbitrary vector field } X,\]

where

\[(1.1b) \quad \bar{X} \overset{\text{def}}{=} FX \text{ and } 'a' \text{ is a complex number.}\]

Then \{F\} is said to give to $M_n$ a hyperbolic differentiable structure, briefly known as HHsu-structure, defined by the equations (1.1) and the manifold $M_n$ is called HHsu-manifold. The equations (1.1) give different structures for different values of \‘a\’[13].
Definition: A structure on an n-dimensional manifold $M$ of class $C^\infty$ given by a non-null tensor field $F$ satisfying

$$F^2 = a' I$$

is called $\pi$-structure or Hsu-structure, where $a$ is a non zero complex constant and $I$ denotes the unit tensor field. Then $M$ is called $\pi$-structure manifold or Hsu-structure manifold[7].

If $a'^2 \neq 0$, it is Hyperbolic $\pi$-structure. If $a'^2 = \pm 1$, it is an almost complex or an almost hyperbolic Product structure, if $a'^2 = \pm i$, it is an almost Product or almost hyperbolic Complex structure and if $a'^2 = 0$, it is an almost tangent or hyperbolic almost tangent structure.

Let the HHsu-structure be endowed with a hermite tensor $g$, such that

(1.2) \[ g(\bar{X}, \bar{Y}) - a^T g(X, Y) = 0. \]

Then $\{F, g\}$ is said to give to $M_a$, a hyperbolic differentiable metric structure and the manifold $M_n$ is called a hyperbolic differentiable metric structure manifold.

Let us put

(1.3) \[ \overset{\text{def}}{\bar{F}(X, Y)} = g(\bar{X}, Y). \]

Then the following equations hold:

(1.4a) \[ \overset{\text{def}}{\bar{F}(X, Y)} = -\overset{\text{def}}{\bar{F}(Y, X)}, \]
i.e. $\bar{F}$ is skew-symmetric in $X$ and $Y$.

(1.4b) \[ \overset{\text{def}}{\bar{F}(\bar{X}, \bar{Y})} = a' \overset{\text{def}}{\bar{F}(X, Y)}, \]
i.e. $\bar{F}$ is hybrid in $X$ and $Y$.

(1.4c) \[ \overset{\text{def}}{\bar{F}(\bar{X}, Y)} = -\overset{\text{def}}{\bar{F}(X, \bar{Y})}. \]

A bilinear function $B$ in HHsu-metric manifold is said to be pure in the two slots, if

(1.5) \[ B(\bar{X}, \bar{Y}) + a' B(X, Y) = 0. \]

It is said to be hybrid in the two slots, if

(1.6) \[ B(\bar{X}, \bar{Y}) - a' B(X, Y) = 0. \]
Let $D$ be a connexion and $X$, $Y$ and $Z$ be $C^\infty$ vector fields in $M_n$, then the function $K$,

defined by

\begin{equation}
K(X, Y, Z) \overset{\text{def}}{=} D_x D_y Z - D_y D_x Z - D_{[x,y]} Z,
\end{equation}

is called the curvature tensor of the connexion $D$.

Let us put

\begin{equation}
\overset{(1.8)}{\dot{K}}(X, Y, Z, T) = g(K(X, Y, Z), T).
\end{equation}

Then $\dot{K}$ is real-valued 4-linear function, called associated curvature tensor or Riemann christoffel curvature tensor of the first kind which satisfies the following properties:

(i) It is skew-symmetric in first two slots:

\begin{equation}
\overset{(1.9)}{\dot{K}}(X, Y, Z, T) = - \overset{(1.8)}{\dot{K}}(Y, X, Z, T).
\end{equation}

(ii) It is skew-symmetric in last two slots:

\begin{equation}
\overset{(1.10)}{\dot{K}}(X, Y, Z, T) = - \overset{(1.8)}{\dot{K}}(X, Y, T, Z).
\end{equation}

(iii) It is symmetric in two pairs of slots:

\begin{equation}
\overset{(1.11)}{\dot{K}}(X, Y, Z, T) = \overset{(1.8)}{\dot{K}}(Z, T, X, Y).
\end{equation}

(iv) It satisfies Bianchi’s first identities:

\begin{equation}
\overset{(1.12)}{\dot{K}}(X, Y, Z, T) + K(Y, Z, X, T) + \overset{(1.8)}{\dot{K}}(Z, X, Y, T) = 0
\end{equation}

(v) It also satisfies Bianchi’s second identities:

\begin{equation}
(D_x \overset{(1.8)}{\dot{K}})(Y, Z, T, U) + (D_y \overset{(1.8)}{\dot{K}})(Z, X, T, U) + (D_z \overset{(1.8)}{\dot{K}})(X, Y, T, U) = 0.
\end{equation}

The tensor defined by

\begin{equation}
\overset{(1.14)}{\text{Ric}}(Y, Z) \overset{\text{def}}{=} (C_1^1 K)(Y, Z),
\end{equation}

is called Ricci tensor, $C_1^1$ being the contraction operator$[5]$.

It is a symmetric tensor of the type (0, 2):

\begin{equation}
\overset{(1.15)}{\text{Ric}}(Y, Z) = \overset{(1.14)}{\text{Ric}}(Z, Y)
\end{equation}
The linear map $r$, defined by

$$(1.16) \quad g(r(X),Y) = g(X, r(Y))$$

is called Ricci map.

The scalar $R$, defined by

$$(1.17) \quad R = (C^i_i r)$$

is called the scalar curvature of $M_n$ at any point $p[6]$.

In the HHsu-metric structure manifold, Pseudo H-Conharmonic Curvature tensor $S^*$, and Pseudo Bochner Curvature tensor $B^*$, are given by

$$(1.18) S^*(X,Y,Z) = K(X,Y,Z) - \frac{a^r}{(n+4)}[a' Ric(Y,Z)X - a' Ric(X,Z)Y - g(X,Z)r(Y)$$

$$+ g(Y,Z)r(X) + Ric(X,\bar{Z})\bar{Y} - Ric(Y,\bar{Z})\bar{X} - 2Ric(\bar{X},Y)\bar{Z}$$

$$+ g(X,\bar{Z})r(\bar{Y}) - g(Y,\bar{Z})r(\bar{X}) - 2g(\bar{X},Y)r(\bar{Z})]$$

$$(1.19) B^*(X,Y,Z) = K(X,Y,Z) - \frac{a^r}{(n+4)}[a' Ric(Y,Z)X - a' Ric(X,Z)Y - g(X,Z)r(Y)$$

$$+ g(Y,Z)r(X) + Ric(X,\bar{Z})\bar{Y} - Ric(Y,\bar{Z})\bar{X} - 2Ric(\bar{X},Y)\bar{Z}$$

$$+ g(X,\bar{Z})r(\bar{Y}) - g(Y,\bar{Z})r(\bar{X}) - 2g(\bar{X},Y)r(\bar{Z})]$$

$$+ \frac{a^r R}{(n+2)(n+4)}[g(Y,Z)X - g(X,Z)Y + g(X,\bar{Z})\bar{Y} - g(Y,\bar{Z})\bar{X} - 2g(\bar{X},Y)\bar{Z}]$$

The associated Pseudo H-Conharmonic curvature tensor $S^*$ and associated Pseudo Bochner curvature tensor $B^*$ are given by[3]:

$$(1.20a) \quad S^*(X,Y,Z,T) = g(S^*(X,Y,Z),T),$$

$$(1.20b) \quad B^*(X,Y,Z,T) = g(B^*(X,Y,Z),T),$$

Consequently,

$$(1.21) S^*(X,Y,Z,T) = K(X,Y,Z,T) - \frac{a^r}{(n+4)}[a' Ric(Y,Z)g(X,T) - a' Ric(X,Z)g(Y,T)$$
\[ + \text{Ric}(X, \bar{Z})g(\bar{Y}, T) - \text{Ric}(Y, \bar{Z})g(\bar{X}, T) - 2\text{Ric}(\bar{X}, Y)g(\bar{Z}, T) - g(X, Z)\text{Ric}(Y, T)\]

\[ + g(Y, Z)\text{Ric}(X, T) - g(Y, \bar{Z})\text{Ric}(\bar{X}, T) - 2g(\bar{X}, Y)\text{Ric}(\bar{Z}, T),\]

\[(1.22) \', (X, Y, Z, T) = 'K(X, Y, Z, T) - \frac{\alpha'}{(n + 4)}[a' \text{Ric}(Y, Z)g(X, T) - a' \text{Ric}(X, Z)g(Y, T)]\]

\[ - g(Y, Z)\text{Ric}(Y, T) + g(Y, Z)\text{Ric}(X, T) + \text{Ric}(X, \bar{Z})g(\bar{Y}, T)\]

\[ - \text{Ric}(\bar{Y}, Z)g(\bar{X}, T) - 2\text{Ric}(\bar{X}, Y)g(\bar{Z}, T) + g(X, \bar{Z})\text{Ric}(\bar{Y}, T)\]

\[ - g(Y, \bar{Z})\text{Ric}(\bar{X}, T) - 2g(\bar{X}, Y)\text{Ric}(\bar{Y}, T)\]

\[ +\frac{a' R}{(n + 2)(n + 4)}[g(Y, Z)g(X, T)]\]

\[ - g(X, Z)g(Y, T) + g(X, \bar{Z})g(\bar{Y}, T) - g(Y, \bar{Z})g(\bar{X}, T) - 2g(\bar{X}, Y)g(\bar{Z}, T).\]

2. **n-Recurrence and n-Recurrence Symmetry of Different Kinds**

**Theorem 2.1.** In the hyperbolic general structure metric manifold \(V_n\), if any two of the following conditions hold for the same \(n\)-recurrence parameter then the third also holds[9]:

(i) It is associated Pseudo H-Conharmonic \((1)\) \(n\)-recurrent,

(ii) It is associated Pseudo Bochner \((1)\) \(n\)-recurrent,

(iii) It is Ricci \(n\)-recurrent,

Provided

\[(2.1) \frac{\alpha'}{(n + 2)(n + 4)}[(\nabla_{n-1} \ldots \nabla_1 R)(U_1, \ldots, U_{n-1})](g(Y, Z)g((\nabla_n F)(\bar{X}, U_n), T)\]

\[- g((\nabla_n F)(\bar{X}, U_n), Z)g(Y, T) + g((\nabla_n F)(\bar{X}, U_n), \bar{Z})g(\bar{Y}, T)\]

\[ + a' g(Y, \bar{Z})g((\nabla_n \bar{X})(U_n), T) + 2a' g((\nabla_n \bar{X})(U_n), Y)g(\bar{Z}, T)\]

\[ + (\nabla_n \nabla_{n-2} \ldots \nabla_1 R)(U_1, \ldots, U_{n-2}, U_n)g(Y, Z)g((\nabla_{n-1} F)(\bar{X}, U_{n-1}), T)\]

\[- g((\nabla_{n-1} F)(\bar{X}, U_{n-1}), Z)g(Y, T) + g((\nabla_{n-1} F)(\bar{X}, U_{n-1}), \bar{Z})g(\bar{Y}, T)\]

\[ + a' g(Y, \bar{Z})g((\nabla_{n-1} \bar{X})(U_{n-1}), T) + 2a' g((\nabla_{n-1} \bar{X})(U_{n-1}), Y)g(\bar{Z}, T)\]
\[\begin{align*}
\sum_{n=1}^{N} \left( \nabla_{n} \cdots \nabla_{1} R \right) \left( U_{1}, U_{3}, \ldots, U_{n} \right) g(Y, Z) & g((\nabla_{2} F)(\vec{X}, U_{2}), T) \\
& - g((\nabla_{2} F)(\vec{X}, U_{2}), Z) g(Y, T) + g((\nabla_{2} F)(\vec{X}, U_{2}), \vec{Z}) g(\vec{Y}, T) \\
& + a^{\prime} g(Y, \vec{Z}) g((\nabla_{2} \vec{X})(U_{2}), T) + 2 a^{\prime} g((\nabla_{2} \vec{X})(U_{2}), Y) g(\vec{Z}, T) \\
& + \left( \nabla_{n} \cdots \nabla_{2} R \right) \left( U_{2}, \ldots, U_{n} \right) g(Y, Z) g((\nabla_{1} F)(\vec{X}, U_{1}), T) \\
& - g((\nabla_{1} F)(\vec{X}, U_{1}), Z) g(Y, T) + g((\nabla_{1} F)(\vec{X}, U_{1}), \vec{Z}) g(\vec{Y}, T) \\
& + a^{\prime} g(Y, \vec{Z}) g((\nabla_{1} \vec{X})(U_{1}), T) + 2 a^{\prime} g((\nabla_{1} \vec{X})(U_{1}), Y) g(\vec{Z}, T) \\
& + \left( \nabla_{n-2} \cdots \nabla_{1} R \right) \left( U_{1}, \ldots, U_{n-2} \right) g(Y, Z) g((\nabla_{n-1} \cdots \nabla_{1} F)(\vec{X}, U_{n-1}, U_{n}), T) \\
& - g((\nabla_{n-1} \cdots \nabla_{1} F)(\vec{X}, U_{n-1}, U_{n}), Z) g(Y, T) + g((\nabla_{n-1} \cdots \nabla_{1} F)(\vec{X}, U_{n-1}, U_{n}), \vec{Z}) g(\vec{Y}, T) \\
& + a^{\prime} g(Y, \vec{Z}) g((\nabla_{n-1} \cdots \nabla_{1} \vec{X})(U_{n-1}, U_{n}), T) + 2 a^{\prime} g((\nabla_{n-1} \cdots \nabla_{1} \vec{X})(U_{n-1}, U_{n}), Y) g(\vec{Z}, T) \\
& + \left( \nabla_{n} R \right) \left( U_{n} \right) g(Y, Z) g((\nabla_{n-1} \cdots \nabla_{1} F)(\vec{X}, U_{1}, \ldots, U_{n-1}), T) \\
& - g((\nabla_{n-1} \cdots \nabla_{1} F)(\vec{X}, U_{1}, \ldots, U_{n-1}), Z) g(Y, T) + g((\nabla_{n-1} \cdots \nabla_{1} F)(\vec{X}, U_{1}, \ldots, U_{n-1}), \vec{Z}) g(\vec{Y}, T) \\
& + a^{\prime} g(Y, \vec{Z}) g((\nabla_{n-1} \cdots \nabla_{1} \vec{X})(U_{1}, \ldots, U_{n-1}), T) + 2 a^{\prime} g((\nabla_{n-1} \cdots \nabla_{1} \vec{X})(U_{1}, \ldots, U_{n-1}), Y) g(\vec{Z}, T) \\
& + (\nabla_{1} R) \left( U_{1} \right) g(Y, Z) g((\nabla_{n} \cdots \nabla_{2} F)(\vec{X}, U_{2}, \ldots, U_{n}), T) - g((\nabla_{n} \cdots \nabla_{2} F)(\vec{X}, U_{2}, \ldots, U_{n}), Z) g(Y, T) \\
& + g((\nabla_{n} \cdots \nabla_{2} F)(\vec{X}, U_{2}, \ldots, U_{n}, \vec{Z}) g(\vec{Y}, T) + a^{\prime} g(Y, \vec{Z}) g((\nabla_{n} \cdots \nabla_{2} \vec{X})(U_{2}, \ldots, U_{n}), T) \\
& + 2 a^{\prime} g((\nabla_{n} \cdots \nabla_{2} \vec{X})(U_{2}, \ldots, U_{n}), Y) g(\vec{Z}, T)) + R \{ g(Y, Z) g((\nabla_{n} \cdots \nabla_{1} F)(\vec{X}, U_{1}, \ldots, U_{n}), T) \\
& - g((\nabla_{n} \cdots \nabla_{1} F)(\vec{X}, U_{1}, \ldots, U_{n}), Z) g(Y, T) + g((\nabla_{n} \cdots \nabla_{1} F)(\vec{X}, U_{1}, \ldots, U_{n}), \vec{Z}) g(\vec{Y}, T) \\
& + a^{\prime} g(Y, \vec{Z}) g((\nabla_{n} \cdots \nabla_{1} \vec{X})(U_{1}, \ldots, U_{n}), T) + 2 a^{\prime} g((\nabla_{n} \cdots \nabla_{1} \vec{X})(U_{1}, \ldots, U_{n}), Y) g(\vec{Z}, T) \} \\
& = 0.
\end{align*}\]
**Proof:** From the equations (1.21) and (1.22), we have

\begin{equation}
(2.2) \quad S^*(X,Y,Z,T) = B^*(X,Y,Z,T) - \frac{a'R}{(n+2)(n+4)}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)]
\end{equation}

+ g(X,\bar{Z})g(\bar{Y},T) - g(Y,\bar{Z})g(\bar{X},T) - 2g(\bar{X},Y)g(\bar{Z},T)]

Barring X in equation (2.2) and using the equation (1.1a) in the resulting equation[10], we get

\begin{equation}
(2.3) \quad S^*(\bar{X},Y,Z,T) = B^*(\bar{X},Y,Z,T) - \frac{a'R}{(n+2)(n+4)}[g(Y,Z)g(\bar{X},T) - g(\bar{X},Z)g(Y,T)]
\end{equation}

- g(\bar{X},\bar{Z})g(\bar{Y},T) + a'g(Y,\bar{Z})g(X,T) + 2a'g(X,Y)g(\bar{Z},T)].

Multiplying the equation (2.3) by $B_n(U_1,.......U_n)$, barring X then using the equation (1.1a) in the resulting equation, we get

\begin{equation}
(2.4) \quad a'B_n(U_1,....U_n)'S^*(X,Y,Z,T) = a'B_n(U_1,....U_n)'B^*(X,Y,Z,T)
\end{equation}

- $\frac{B_n(U_1,..........U_n)R}{(n+2)(n+4)}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)]$

+ g(X,\bar{Z})g(\bar{Y},T) - g(Y,\bar{Z})g(\bar{X},T) - 2g(\bar{X},Y)g(\bar{Z},T)]

Differentiating the equation (2.3) with respect to $U_1,...,U_n$, using the equation (2.3), then barring X and subtracting the equation (2.4) from the resulting equation[15], we get

\begin{equation}
(2.5) \quad a'(\nabla_{n}...\nabla_1'S^*)(X,Y,Z,T,U_1,....U_n)
\end{equation}

- $(\nabla_{n-1}...\nabla_1'S^*)((\nabla_{n}F)(\bar{X},U_n),Y,Z,T,U_1,....,U_{n-1}............)
\end{equation}

- $(\nabla_{n}...\nabla_3\nabla_1'S^*)((\nabla_{2}F)(\bar{X},U_2),Y,Z,T,U_1,U_3,....,U_n)............)
\end{equation}

- $(\nabla_{n}...\nabla_3\nabla_2'S^*)((\nabla_{1}F)(\bar{X},U_1),Y,Z,T,U_2,U_3,....,U_n)............)
\end{equation}

- $(\nabla_{n-2}...\nabla_1'S^*)((\nabla_{n}\nabla_{n-1}F)(\bar{X},U_{n-1},U_n),Y,Z,T,U_1,....,U_{n-2})............)
\end{equation}

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\[-(\nabla_n \cdots \nabla_3 'S')((\nabla_2 \nabla_1 F)(\vec{X}, U_1, U_2), Y, Z, T, U_3, \ldots, U_n)\]
\[ + a' g(Y, Z) g((\nabla_n X)(U_n), T) + 2a' g((\nabla_n X)(U_n), Y) g(\bar{Z}, T) \]

\[ + (\nabla_n \nabla_{n-2} \ldots \nabla_1 R)(U_1, \ldots, U_{n-2}, U_n) g(Y, Z) g((\nabla_{n-1} F)(\bar{X}, U_{n-1}), T) \]

\[ - g((\nabla_{n-1} F)(\bar{X}, U_{n-1}), Z) g(Y, T) + g((\nabla_{n-1} F)(\bar{X}, U_{n-1}), \bar{Z}) g(\bar{Y}, T) \]

\[ + a' g(Y, Z) g((\nabla_{n-1} \bar{X})(U_{n-1}), T) + 2a' g((\nabla_{n-1} \bar{X})(U_{n-1}), Y) g(\bar{Z}, T) \]

\[ + (\nabla_n \ldots \nabla_2 R)(U_2, \ldots, U_n) g(Y, Z) g((\nabla_1 F)(\bar{X}, U_1), T) \]

\[ - g((\nabla_1 F)(\bar{X}, U_1), Z) g(Y, T) + g((\nabla_1 F)(\bar{X}, U_1), \bar{Z}) g(\bar{Y}, T) \]

\[ + a' g(Y, Z) g((\nabla_{1} \bar{X})(U_1), T) + 2a' g((\nabla_{1} \bar{X})(U_1), Y) g(\bar{Z}, T) \]

\[ + (\nabla_{n-2} \ldots \nabla_1 R)(U_1, \ldots, U_{n-2}) g(Y, Z) g((\nabla_{n-1} \nabla_{n-1} F)(\bar{X}, U_{n-1}, U_n), T) \]

\[ - g((\nabla_n \nabla_{n-1} F)(\bar{X}, U_{n-1}, U_n), Z) g(Y, T) + g((\nabla_n \nabla_{n-1} F)(\bar{X}, U_{n-1}, U_n), \bar{Z}) g(\bar{Y}, T) \]

\[ + a' g(Y, Z) g((\nabla_{n-1} \nabla_{n-1} \bar{X})(U_{n-1}, U_n), T) + 2a' g((\nabla_{n-1} \nabla_{n-1} \bar{X})(U_{n-1}, U_n), Y) g(\bar{Z}, T) \]

\[ + (\nabla_n \ldots \nabla_3 R)(U_3, \ldots, U_n) g(Y, Z) g((\nabla_2 \nabla_1 F)(\bar{X}, U_1, U_2), T) \]

\[ - g((\nabla_2 \nabla_1 F)(\bar{X}, U_1, U_2), Z) g(Y, T) + g((\nabla_2 \nabla_1 F)(\bar{X}, U_1, U_2), \bar{Z}) g(\bar{Y}, T) \]

\[ + a' g(Y, Z) g((\nabla_2 \nabla_1 \bar{X})(U_1, U_2), T) + 2a' g((\nabla_2 \nabla_1 \bar{X})(U_1, U_2), Y) g(\bar{Z}, T) \]

\[ + (\nabla_n R)(U_n) g(Y, Z) g((\nabla_{n-1} \ldots \nabla_1 F)(\bar{X}, U_1, \ldots, U_{n-1}), T) \]

\[ - g((\nabla_{n-1} \ldots \nabla_1 F)(\bar{X}, U_1, \ldots, U_{n-1}), Z) g(Y, T) + g((\nabla_{n-1} \ldots \nabla_1 F)(\bar{X}, U_1, \ldots, U_{n-1}), \bar{Z}) g(\bar{Y}, T) \]

\[ + a' g(Y, \bar{Z}) g((\nabla_{n-1} \ldots \nabla_1 \bar{X})(U_1, \ldots, U_{n-1}), T) + 2a' g((\nabla_{n-1} \ldots \nabla_1 \bar{X})(U_1, \ldots, U_{n-1}), Y) g(\bar{Z}, T) \]

\[ + (\nabla_1 R)(U_1) \{ g(Y, Z) g((\nabla_{n-2} \ldots \nabla_2 F)(\bar{X}, U_2, \ldots, U_n), T) - g((\nabla_{n-2} \ldots \nabla_2 F)(\bar{X}, U_2, \ldots, U_n), Z) g(Y, T) \]
$$+2a^\prime g((\nabla_n\ldots\nabla_2 X)(U_2,\ldots,U_n),Y)g(Z,T)\} + R[g(Y,Z)g((\nabla_n\ldots\nabla_1 F)(X,U_1,\ldots,U_n),T)$$

$$-g((\nabla_n\ldots\nabla_1 F)(X,U_1,\ldots,U_n),Z)g(Y,T) + g((\nabla_n\ldots\nabla_1 F)(X,U_1,\ldots,U_n),Z)g(Y,T)$$

$$+a^\prime g(Y,Z)g((\nabla_n\ldots\nabla_1 X)(U_1,\ldots,U_n),T) + 2a^\prime g((\nabla_n\ldots\nabla_1 X)(U_1,\ldots,U_n),Y)g(Z,T)\}$$

Let the hyperbolic general structure metric manifold is associated Pseudo H-Conharmonic (1) n-recurrent and associated Pseudo Bochner (1) n-recurrent for the same n-recurrence parameter and the equation (2.1) is satisfied[4], then from the equation (2.5), we have

$$\frac{1}{(n+2)(n+4)}\{(\nabla_n\ldots\nabla_1 R)(U_1,\ldots,U_n) - RB_n(U_1,\ldots,U_n)\}g(Y,Z)g(X,T)$$

$$-g(X,Z)g(Y,T) + g(X,Z)g(Y,T) - g(Y,Z)g(X,T) - 2g(X,Y)g(Z,T) = 0$$

or

$$(\nabla_n\ldots\nabla_1 R)(U_1,\ldots,U_n) = RB_n(U_1,\ldots,U_n).$$

Which shows that the manifold is Ricci n-recurrent.

Similarly it can be shown that if the hyperbolic general structure metric manifold is either associated Pseudo H-Conharmonic (1) n-recurrent and Ricci n-recurrent or associated Pseudo Bochner (1) n-recurrent and Ricci n-recurrent then it is either associated Pseudo Bochner (1) n-recurrent or associated Pseudo H-Conharmonic (1) n-recurrent for the same n-recurrence parameter provided the equation (2.1) is satisfied.

**Theorem 2.2.** In the hyperbolic general structure metric manifold $M_n$, if any two of the following conditions hold for the same n-recurrence parameter then the third also holds[12]:

(i) It is associated Pseudo H-Conharmonic(1) n-recurrent symmetric,

(ii) It is associated Pseudo Bochner (1) n-recurrent symmetric,

(iii) It is Ricci n-recurrence symmetric,

Provided

$$a^\prime \frac{1}{(n+2)(n+4)}\{(\nabla_n\ldots\nabla_1 R)(U_1,\ldots,U_n)\}g(Y,Z)g((\nabla_n F)(X,U_n),T)$$

$$-g((\nabla_n F)(X,U_n),Z)g(Y,T) + g((\nabla_n F)(X,U_n),Z)g(Y,T)$$

$$+a^\prime g(Y,Z)g((\nabla_n X)(U_n),T) + 2a^\prime g((\nabla_n X)(U_n),Y)g(Z,T)\}$$
\[\begin{align*}
&+\left(\nabla_n \cdots \nabla_{n-1} \nabla_1 R\right) (U_1, \ldots, U_{n-2}, U_n) g(Y, Z) g((\nabla_{n-1} F)(\bar{X}, U_{n-1}), T) \\
&\quad - g((\nabla_{n-1} F)(\bar{X}, U_{n-1}), Z) g(Y, T) + g((\nabla_{n-1} F)(\bar{X}, U_{n-1}), \bar{Z}) g(\bar{Y}, T) \\
&\quad + a^\prime g(Y, \bar{Z}) g((\nabla_{n-1} \bar{X})(U_{n-1}), T) + 2a^\prime g((\nabla_{n-1} \bar{X})(U_{n-1}), Y) g(\bar{Z}, T) \}
\end{align*}\]

\[\begin{align*}
&+\left(\nabla_n \cdots \nabla_3 \nabla_1 R\right) (U_1, U_3, \ldots, U_n) g(Y, Z) g((\nabla_2 F)(\bar{X}, U_2), T) \\
&\quad - g((\nabla_2 F)(\bar{X}, U_2), Z) g(Y, T) + g((\nabla_2 F)(\bar{X}, U_2), \bar{Z}) g(\bar{Y}, T) \\
&\quad + a^\prime g(Y, \bar{Z}) g((\nabla_2 \bar{X})(U_2), T) + 2a^\prime g((\nabla_2 \bar{X})(U_2), Y) g(\bar{Z}, T) \}
\end{align*}\]

\[\begin{align*}
&+\left(\nabla_{n-2} \cdots \nabla_1 R\right) (U_1, \ldots, U_{n-2}) g(Y, Z) g((\nabla_{n-1} \nabla_{n-2} F)(\bar{X}, U_{n-2}, U_n), T) \\
&\quad - g((\nabla_{n-1} \nabla_{n-2} F)(\bar{X}, U_{n-2}, U_n), Z) g(Y, T) + g((\nabla_{n-1} \nabla_{n-2} F)(\bar{X}, U_{n-2}, U_n), \bar{Z}) g(\bar{Y}, T) \\
&\quad + a^\prime g(Y, \bar{Z}) g((\nabla_{n-1} \nabla_{n-2} \bar{X})(U_{n-2}, U_n), T) + 2a^\prime g((\nabla_{n-1} \nabla_{n-2} \bar{X})(U_{n-2}, U_n), Y) g(\bar{Z}, T) \}
\end{align*}\]

\[\begin{align*}
&+\left(\nabla_n R\right) (U_n) g(Y, Z) g((\nabla_{n-1} \cdots \nabla_1 F)(\bar{X}, U_1, \ldots, U_{n-1}), T) \\
&\quad - g((\nabla_{n-1} \cdots \nabla_1 F)(\bar{X}, U_1, \ldots, U_{n-1}), Z) g(Y, T) + g((\nabla_{n-1} \cdots \nabla_1 F)(\bar{X}, U_1, \ldots, U_{n-1}), \bar{Z}) g(\bar{Y}, T) \\
&\quad + a^\prime g(Y, \bar{Z}) g((\nabla_{n-1} \cdots \nabla_1 \bar{X})(U_1, \ldots, U_{n-1}, T) + 2a^\prime g((\nabla_{n-1} \cdots \nabla_1 \bar{X})(U_1, \ldots, U_{n-1}, Y) g(\bar{Z}, T)) \}
\end{align*}\]

\[\begin{align*}
&(\nabla_1 R)(U_1) g(Y, Z) g((\nabla_{n-1} \cdots \nabla_2 F)(\bar{X}, U_2, \ldots, U_n), T) - g((\nabla_{n-1} \cdots \nabla_2 F)(\bar{X}, U_2, \ldots, U_n), Z) g(Y, T) \\
&\quad + g((\nabla_{n-1} \cdots \nabla_2 F)(\bar{X}, U_2, \ldots, U_n), \bar{Z}) g(\bar{Y}, T) + a^\prime g(Y, \bar{Z}) g((\nabla_{n-1} \cdots \nabla_2 \bar{X})(U_2, \ldots, U_n), T) \\
&\quad + 2a^\prime g((\nabla_{n-1} \cdots \nabla_2 \bar{X})(U_2, \ldots, U_n), Y) g(\bar{Z}, T) + R g(Y, Z) g((\nabla_{n-1} \cdots \nabla_1 F)(\bar{X}, U_1, \ldots, U_n), T) \\
&\quad + 2a^\prime g((\nabla_{n-1} \cdots \nabla_2 F)(\bar{X}, U_2, \ldots, U_n), \bar{Z}) g(\bar{Y}, T) - g((\nabla_{n-1} \cdots \nabla_1 F)(\bar{X}, U_1, \ldots, U_n), \bar{Z}) g(\bar{Y}, T)
\end{align*}\]
+ \alpha' g(Y, \overline{Z}) g((\nabla_n ... \nabla_1 \overline{X})(U_1, ..., U_n), T) + 2\alpha' g((\nabla_n ... \nabla_1 \overline{X})(U_1, ..., U_n), Y) g(\overline{Z}, T) = 0.

**Proof:** Let the hyperbolic Hsu-structure metric manifold is associated Pseudo H-Conharmonic (1) n-recurrent symmetric and associated Pseudo Bochner (1) n-recurrent symmetric then from the equation (2.5), we have

\begin{equation}
(2.8) \left[ \nabla_n ... \nabla_1 R \right](U_1, ..., U_n) g(Y, Z) g(X, T) - g(X, Z) g(Y, T) + g(X, \overline{Z}) g(\overline{Y}, T)
- g(Y, \overline{Z}) g(\overline{X}, T) - 2g(\overline{X}, Y) g(\overline{Z}, T) = 0.
\end{equation}

or

\begin{equation}
(\nabla_n ... ... \nabla_1 R)(U_1, ..., U_n) = 0.
\end{equation}

provided the equation (2.7) is satisfied, which shows that the manifold is Ricci n-recurrent symmetric[14].

Similarly, it can be shown that if the hyperbolic Hsu-structure metric manifold is either associated Pseudo H-Conharmonic (1) n-recurrent symmetric and Ricci n-recurrent symmetric or associated Pseudo Bochner (1) n-recurrent symmetric and Ricci n-recurrent symmetric or associated Pseudo H-Conharmonic (1) n-recurrent symmetric for the same n-recurrence parameter provided the equation (2.7) is satisfied.

**Note 2.1.** Theorems of the type (2.1) and (2.2) can also be stated and proved taking (2), (3), (4), (12), (13), (14), (23), (34), (123), (124), (134), (234), or (1234) n-recurrent and (2), (3), (4), (12), (13), (14), (23), (34), (123), (124), (134), (234), or (1234) n-recurrent symmetric hyperbolic Hsu-structure metric manifold.

**References:**

2. S. S. Chen (1968), *Topics in differential geometry*, Institute for advanced study, lecture notes, 1951.


