Some Fixed Point Results In Partial $b_v(s)$-Metric Spaces

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Abstract: In this manuscript, we established and proved some results on fixed point in the framework of partial $b_v(s)$- metric space. The obtained results generalize the existing results in the literature.

INTRODUCTION

Fixed point theory is one of the strong tools of modern mathematics. The theorems that are related with fixed points and there properties are known as fixed point theorems. This theory is the wonderful combination of analysis, topology and geometry. Fixed point theorem has got application in the various fields such as mathematics engineering, physics, economics, game theory, biology, chemistry etc. In mathematics fixed points are an important part of nonlinear functional analysis. The study of fixed points has been at the centre of energetic research activity in the last decades where the mappings satisfying certain contractive conditions in different abstract spaces. The Banach mapping contraction principle is one of the incipient and basic results in this direction. In most of the problems whenever the solution exists fixed point will also exist naturally. Therefore the existence of fixed point is very important in various fields of mathematics and other sciences. Fixed point theorems provide conditions under which maps have solutions. The theory of fixed points thus a great and delighted combination of analysis (pure and applied). In 1994, Matthews [10] gave the prospect of partial metric space .The typical separation was changed by incomplete measurement in partial metric space with an energizing property 'positive self-separation of points'. In this space the assembly of a grouping was characterized in such a way, that the limit of the convergent sequence need not to be special. In partial metric space Matthews gave the guarantee of the legitimacy of Banach fixed point theorem and proved that it can be used for the verification of programmes. After that Matthews results were generalized by several authors. Partial metric space thought was further generalized by O'Neill by acknowledging negative distances. O'Neill defined a partial metric which is known as dualistic partial metric. By neglecting the concept of small self-distance condition. Hickmann partial metric is known as weak partial metric. Wardowski displayed another idea of -constriction and demonstrated a fixed-point hypothesis which sums up the Banach fixed point hypothesis in an entirely unexpected manner than the hypotheses that are as of now existing in the writing on complete measurement spaces.

Definition 1 Partial-metric space [10] Let $d_p: X \times X \to [0, \infty)$be a function defined on a non-empty set X satisfying the following properties.

$(p1)$ $\mu_1 = \mu_2$ iff $d_p(\mu_1, \mu_1) = d_p(\mu_1, \mu_2) = d_p(\mu_2, \mu_2)$ for all $\mu_1, \mu_2 \in X$

$(p2)$ $d_p(\mu_1, \mu_2) \leq d_p(\mu_1)$for all $\mu_1, \mu_2 \in X$

$(p3)$ $d_p(\mu_1, \mu_2) = d_p(\mu_2, \mu_1)$ for all $\mu, \mu_2 \in X$

$(p4)$ $d_p(\mu_1, \mu_2) \leq d_p(\mu_1, \mu_3) + d_p(\mu_3, \mu_2) - d_p(\mu_3, \mu_3)$

Then $d_p$is called partial metric on X and $(X, d_p)$is called partial metric space.

Definition 2 Partial b-metric space [see 2] Define a function $b: Y \times Y \to [0, \infty)$, on a non-void set Y .Then it qualifies as a partial b metric on Y if it holds the following properties:

(1) $\mu_1 = \mu_2 \; iff \; b(\mu_1, \mu_1) = b(\mu_1, \mu_2) = b(\mu_2, \mu_2) \; for \; all \; \mu_1, \mu_2 \in Y$

(2) $b(\mu_1, \mu_1) \leq b(\mu_1, \mu_2)$ for all $\forall \mu_1, \mu_2 \in Y$;

(3) $b(\mu_1, \mu_2) = b(\mu_2, \mu_1)$ for all $\mu_1, \mu_2 \in Y$;

There is $s \geq 1$ s.t $\forall \mu_1, \mu_2, u_1 \in X, b(\mu_1, \mu_2) \leq s[b(\mu_1, u_1) + b(u_1, \mu_2)] - b(u_1, u_2)$.

And the pair (X, b) is said to be a partial b-metric space.

Definition 3 $b_v(s)$ -metric space [see 2] Let $b_v: X \times X \to [0, \infty)$ be a function defined on a non-void set $X$and $v \in \mathbb{N}$ such that if for all distinct points $u_1, u_2, \ldots, u_v \in X - \{\mu_1, \mu_2\}$ the following hold:

(1) $b_v(\mu_1, \mu_2)$ = 0 iff $\mu_1 = \mu_2 \;\; \mu_1, \mu_2 \in X$

(2) $b_v(\mu_1, \mu_2) = b_v(\mu_2, \mu_1)$;

(3) There is $s \epsilon \mathbb{R}$ with s $\geq$ 1 such that $b_v(\mu_1, \mu_2) \; \leq s[b_v(\mu_1, u_1)$ + $b_v(u_1, u_2)$ + … + $b_v(u_v, \mu_2)]$.

Then b_v is called a $b_v(s)$- matrix on X, and (X, b_v) is called a $b_v(s) -$ metric space with a coefficient s.

Definition 4 PARTIAL $b_v(s)$ metric space [2] Let pb_v : X × X→ $[0, \infty)$ be a function defined on a non-void set $X$and v ∈ ℕ such that if for all $\mu_1, \mu_2 \in X$ and for all different points $u_1, u_2, \ldots, u_v \in \; X \backslash \{\mu_1, \mu_2\}$, the following hold:

(1) $\mu_1 = \mu_2$ Iff $pb_v(\mu_1, \mu_1)$ =$pb_v(\mu_1, \mu_2)$ =$pb_v(\mu_2, \mu_2)$;

(2) $pb_v(\mu_1, \mu_1) \leq pb_v(\mu_1, \mu_2)$;

(3) $pb_v(\mu_1, \mu_2) = pb_v(\mu_2, \mu_1)$

(4) There is $s \epsilon \mathbb{R}$ with s$\geq$ 1 such that

$pb_v(\mu_1, \mu_2) \leq s[pb_v(\mu_1, \; u_1) + pb_v(u_1, u_2) + \cdots + pb_v(u_v, \mu_2)] - \sum_{i=1}^{n} pb_v(u_i, u_i)$

Then (X, pb_v) is called a partial b_v(s)-metric space with a coefficient s and with a partial $b_v(s)$ metric denoted by pb_v.

Example 5 (see 2) Let X = {i, j, k, l} and let us define a function pb_v : X × $X \to \; \mathbb{R}_+$by:

$$pb_v(\mu_1, \mu_2) = \begin{cases} 0, & if \; \mu_1 = \mu_2 = \text{i} \\ 2, & if \; \mu_1, \mu_2 \in \{i, j\}, \mu_1 \neq \mu_2; \\ 1, & otherwise \end{cases}$$

For all $\mu_1, \mu_2 \in X$.

Then the space X together with the metric pb_v qualifies as $b_2 \; (\frac{4}{3})$ which is partial in nature. Which is not a metric space nor a $b_2(1)$ metric space which is partial in nature, because $pb_v(j, j) \neq 0$ and

$$pb_v(i, j) = 2 > 1$$

$$= pb_v(i, k) + pb_v(k, l) + pb_v(l, j) - pb_v(k, k) + pb_v(l, l), \text{ respectively.}$$

DEFNITION 6 (see 2) Suppose Y be a space and pb_v be a metric defined on it. Then the pair (Y, pb_v) qualifies as a $b_v(s)$-metric space which is partial in nature with coefficient $s \geq 1$. Suppose $\{\mu_n\} \in$ Y in (Y, pb_v). Then,

(a) $\{\mu_n\}$ is said to converge to x with respect to τ_{pb_v}if and only if $\lim_{n \to \infty} pb_v(\mu_n, \mu) = pb_v(\mu, \mu)$. Moreover, μ is called limit point of $\{\mu_n\}$;

(b) $\{x_n\}$ is called Cauchy if $\lim_{n,m \to \infty} pb_v(\mu_n, \mu_m)$ exists and is finite;

(c) $\{\mu_n\}$ is a Cauchy sequence in X, if there is $x \in X$ s.t $\lim_{n,m \to \infty} pb_v(\mu_n, \mu_m) = \lim_{n \to \infty} pb_v(\mu_n, \mu) = pb_v(\mu, \mu)$, then (X, pb_v) is a partial $b_v(s)$ metric space having the completeness property.

Definition 7 (see 2) An open ball with centre at $\mu_1 \in X$ and radius $\epsilon > 0$ of a partial $b_v(s)$ −metric spaces can be written as:

$$B_{pb_v}(\mu_1, \epsilon) \coloneqq \{\mu_2 \in X: pb_v(\mu_1, \mu_2) < pb_v(\mu_1, \mu_1) + \epsilon\}$$

In partial $b_v(s)-$metric space an open ball may be empety as well.For example, if $pb_v(\mu_1, \mu_1) > 0$, then$B_{pb_v}\big(\mu_1, pb_v(\mu_1, \mu_1)\big) = \emptyset$.

Proposition 8 (see 2) Suppose X be a non-void set and pb_v be a metric defined on it, then the pair (X, pb_v) qualifies as a $b_v(s)$ metric spaces which is partial in nature with $s \geq 1$ possessing let property and let B be the collection of all open balls in (X, pb_v), then B will qualify as a basis for topology on X.

Proposition 9 (see 2). Suppose we have a metric space of partial $b_v(s)$ type denoted by $(X, \mathrm{p}b_v)$ and let $\lambda \in [0, \infty)$, then the pair (X, d) is a metric space of partial $b_v(s)$ type where

$$d(\mu_1, \mu_2) = \lambda + pb_v(\mu_1, \mu_2) \quad \forall \mu_1, \mu_2 \in X$$

Proposition 10 (see 3). Suppose b_v denote $b_v(s) -$ metric with coefficient $s \geq 1$ and d_p be a partial metric defined on a non-void set X. If $d: X \times X \to [0, \infty)$ is given by

$$d(\mu_1, \mu_2) = d_p(\mu_1, \mu_2) + b_v(\mu_1, \mu_2)$$

For all $\mu_1, \mu_2 \in X$, then the pair (X, d) is a $b_v(s)$ −metric space, which is partial in nature.

Lemma 11 [13] Let $T: Y \to Y$ be a function and let (Y, pb_v) be a $b_v(s)$ −metric space which is partial in nature with $s \geq 1$. If $\{\mu_n\} \in X$ given by $T\mu_n = \mu_{n+1}$ for all $n \geq 0$ with $\mu_n \neq \mu_{n+1}$. Let $k \in [0,1)$ such that for all $n \in \mathbb{N}$

$$pb_v(\mu_{n+1}, \mu_n) \leq kpb_v(\mu_n, \mu_{n-1}).$$

Then, either T has a fixed point or $\mu_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$.

Lemma 12 [13] Let (X, pb_v) is a $b_v(s)$ −metric space which is partial in nature with $s \geq 1$ and $\{\mu_n\} \in X$ s.t $\forall\ n \geq 0,\ \mu_n \neq \mu_{n+1}$. Let $k \in [0,1)$ and $\alpha, \beta, \tau, \delta \in \mathbb{R}^+$ such that for all $n, m \in \mathbb{N}$.

$$pb_v(\mu_n, \mu_m) \leq kpb_v(\mu_{n-1}, \mu_{m-1}) + (\alpha + s\tau)k^n + (\beta + s\delta)k^m.$$

Then $\{\mu_n\}$ is a Cauchy.

Theorem 13. [13] Let (X,pb_v) be a partial $b_v(s)$ −metric space having the completeness property with s≥ 1 and T be a self -map fulfilling the below condition:

$$pb_v(T_{\mu_1}, T_y) \leq \lambda pb_v(\mu_1, y)$$

$\forall\ \mu_1, y \in X$, where $\lambda \in [0,1)$ and $s \geq 1$. Then T possesses a fixed point $u \in X$ which is unique and $pb_v(u, u) = 0$.

Theorem 14 [13] Assume (X, pb_v) be a partial $b_v(s)$ −metric space having the completeness property with $s \geq 1$, and T be a self- mapping on X which fulfils the below condition

$$pb_v\big(T_{\mu_1}, T_{\mu_2}\big) \leq \lambda_1 pb_v(\mu_1, \mu_2) + \lambda_2 pb_v\big(\mu_1, T_{\mu_1}\big) + \lambda_3 pb_v(\mu_2, T_{\mu_2})$$

$\forall\, \mu_1, \mu_2 \in X$, Where λ_i are positive real and $\lambda_i \in \mathbb{R}$ with $\sum_{i=1}^{n} \lambda_i < 1$ and $min\{\lambda_2, \lambda_3\} < 1$. Then T possesses a fixed point $u \in X$ which is unique and $pb_v(u, u) = 0$.

Example 15 [13] Suppose $Y = \{0,1,2,3,6\}$ and suppose $pb_v: Y \times Y \to [0, \infty)$ which is given by:

$$pb_v(\mu_1, \mu_2) = \begin{cases} \dfrac{1}{2}, & if\ \ \mu_1 = \mu_2 = 1 \\ \mu_1, & if\ \ \mu_1 = \mu_2 \neq 1 \\ 2(\mu_1 - \mu_2)^2 + \mu_1 + \mu_2, & if\ \ \mu_1, \mu_2 \in \{1,3\} \\ |\mu_1 - \mu_2| & otherwise \end{cases}$$

Then (Y, pb_v) is a partial $b_3(2) -$ metric space having the completeness property.

Suppose T be a self- map on X which is given by:

$$T(\mu) = \begin{cases} 0 & if \quad \mu = 6 \\ 1 & if \quad \mu \in \{0,1,2,3\} \end{cases}$$

$\forall\ \mu_1, \mu_2 \in X$. Then T fulfils all the prerequisites of the theorems (2.1) and (2.2) and hence the existence of fixed point $\mu = 1$ which is unique with $k \in [\frac{1}{3}, 1)$ and $\lambda_1 = \lambda_2 = \frac{1}{6}$, $\lambda_3 = \frac{1}{3}$ respectively.

Corollary 16 [13] Suppose (Y, pb_v) be a partial metric space of $b_v(s)$ type having the completeness property with s≥ 1 and T be a self-mapping defined on X fulfilling the below condition:

$$pb_v(T_{\mu_1}, T_{\mu_2}) \leq \gamma(pb_v(\mu_1, T_{\mu_1}) + pb_v(\mu_2, T_{\mu_2}))$$

$\forall\ \mu_1, \mu_2 \in X$, where $k \in [0,1)$. Then T possesses a fixed point $u \in X$ which is unique a unique and $pb_v(u, u) = 0$

Main Section

In this section, we have proved some fixed point results in the framework of partial $b_v(s)$-metric spaces.

Theorem 17. Let (X, pb_v) be a complete partial $b_v(s)$ $-$metric space with coefficient $s \geq 1$, and $T: X \to X$ be a self-mapping satisfying the following conditions:

$pb_v(T\mu_1, T\mu_2) \leq apb_v(\mu_1, T\mu_2) + bpb_v(\mu_2, T\mu_1) + cpb_v(\mu_1, \mu_2) \quad \forall\, \mu_1, \mu_2 \in X,$ ... (1)

where a, b, c are non-negative real numbers with $a + b + c < 1$ and $\min\{a, b\} < 1$. Then T has a unique fixed point $u \in X$ and $pb_v(u, u) = 0$.

Proof. First we shall prove the existence of a fixed point and then its uniqueness.

Existence. Let μ_0 be an arbitrary number and $\{\mu_n\}$ be a sequence in X defined by

$$T\mu_n = x_{n+1} \quad \forall\ n \geq 0$$

By (1), we have

$$\begin{aligned} pb_v(\mu_{n+1}, \mu_n) &\leq apb_v(\mu_n, T\mu_{n-1}) + bpb_v(\mu_{n-1}, T\mu_n) + cpb_v(\mu_n, \mu_{n-1}) \\ &= apb_v(\mu_n, \mu_n) + bpb_v(\mu_{n-1}, \mu_{n+1}) + cpb_v(\mu_n, \mu_{n-1}) \\ &= apb_v(\mu_{n+1}, \mu_n) + bpb_v(\mu_{n-1}, \mu_{n+1}) + cpb_v(\mu_{n-1}, \mu_{n+1}) \\ &= hpb_v(\mu_n, \mu_{n-1}) \text{ , where } h = \frac{c+a}{1-b} < 1. \qquad \text{... (2)} \end{aligned}$$

From equation (2), it follows that

$$pb_v(\mu_{n+1}, \mu_n) \leq h^n pb_v(\mu_1, \mu_0) \quad \forall\ n \in \mathbb{N}$$

If $\mu_n = \mu_{n+1}$, then μ_n is a fixed point of T and we have nothing to prove.

Now, we shall suppose that $\mu_n \neq \mu_{n+1}\ \forall\, n \geq 0$. Then, it follows from lemma (12) $\mu_n \neq \mu_m$ for all distinct $n, m \in N$. Moreover from equation (2), we have

$$\begin{aligned} pb_v(\mu_n, \mu_m) &\leq apb_v(\mu_{n-1}, T\mu_{m-1}) + bpb_v(\mu_{m-1}, T\mu_{n-1}) + cpb_v(\mu_{n-1}, \mu_{m-1}) \\ &\leq h^{n-1}apb_v(\mu_1, \mu_0) + \ h^{m-1}bpb_v(\mu_1, \mu_0) + cpb_v(\mu_{n-1}, \mu_{m-1}) \end{aligned}$$

Taking $K = \max\{c, h\}$, $\alpha = ah^{-1}pb_v(\mu_1, x_0)$, $\beta = bh^{-1}pb_v(\mu_1, x_0)$ and $\zeta = \delta = 0$.

It follows from lemma (12) that $\{\mu_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $\mu^\star \in X$ such that

$$\lim_{n,m\to\infty} pb_v\,(\mu_n, \mu_m) = \lim_{n\to\infty} pb_v(\mu_n, \mu^\star) = pb_v\,(\mu^\star, \mu^\star) = 0. \qquad \text{... (3)}$$

We now show that $\mu^\star$ is a fixed point of T.

$$\begin{aligned} pb_v(\mu^\star, T\mu^\star) &\leq s[pb_v(\mu^\star, \mu_n) + pb_v(\mu_n, \mu_{n+1}) + \cdots + pb_v(\mu_{n+v-1}, \mu_{n+v}) \\ &\quad + pb_v(\mu_{n+v}, T\mu^\star)] - \sum_{k=1}^{v} pb_v(\mu_{n+k}, \mu_{n+k}) \\ &\leq s[pb_v(\mu^\star, \mu_n) + pb_v(\mu_n, \mu_{n+1}) + \cdots + pb_v(\mu_{n+v-1}, \mu_{n+v}) + pb_v(T\mu_{n+v-1}, T\mu^\star)] \\ &\quad - \sum_{k=1}^{v} pb_v\,(\mu_{n+k}, \mu_{n+k}) \end{aligned}$$

$$\leq s[pb_v(\mu^\star, \mu_n) + pb_v(\mu_n, \mu_{n+1}) + \cdots + pb_v(\mu_{n+v-1}, \mu_{n+v}) + cpb_v(\mu_{n+v-1}, \mu^\star) + apb_v(\mu_{n+v-1}, \mu_{n+v}) + bpb_v(\mu^\star, T\mu^\star) - \sum_{k=1}^{v} pb_v(\mu_{n+k}, \mu_{n+k})$$

$$\leq s[pb_v(\mu^\star, \mu_n) + pb_v(\mu_n, \mu_{n+1}) + \cdots + pb_v(\mu_{n+v-1}, \mu_{n+v}) + apb_v(\mu_{n+v-1}, \mu_{n+v}) + bpb_v(\mu^\star, T\mu^\star) + cpb_v(\mu_{n+v-1}, \mu^\star) \quad .... (4)$$

From (3) as $n \to \infty$ in (4), we get

$$(1-b)pb_v(\mu^\star, T\mu^\star) \leq 0$$

Similarly, we can show that

$(1-a)pb_v(T\mu^\star, \mu^\star) \leq 0$, with $\min\{a, b\} < 1$

$$pb_v(\mu^\star, T\mu^\star) = 0.$$

Therefore, $T\mu^\star = \mu^\star$

Hence, $\mu^\star$ is a fixed point of T.

Uniqueness.

$$\begin{aligned} pb_v(\mu^\star, \mu^\star) &= pb_v(T\mu^\star, T\mu^\star) \\ &\leq apb_v(\mu^\star, T\mu^\star) + bpb_v(\mu^\star, T\mu^\star) + cpb_v(\mu^\star, \mu^\star) \\ &= apb_v(\mu^\star, \mu^\star) + bpb_v(\mu^\star, \mu^\star) + cpb_v(\mu^\star, \mu^\star) \\ &= (a+b+c)pb_v(\mu^\star, \mu^\star) \\ &< m\, pb_v(\mu^\star, \mu^\star), \end{aligned}$$

which is a contradiction. Hence, $pb_v(\mu^\star, \mu^\star) = 0$.

Let $\mu^\star, u \in X$ be the two distinct points $i.e\ \mu^\star \neq u\ s.t\ T\mu^\star = \mu^\star$ and $Tu = u$. Then, it follows from equation (1) that we have

$$\begin{aligned} pb_v(\mu^\star, u) &= pb_v(T\mu^\star, Tu) \\ &\leq a\, pb_v(\mu^\star, Tu) + b\, pb_v(u, T\mu^\star) + c\, pb_v(\mu^\star, u) \\ &= a\, pb_v(\mu^\star, u) + b\, pb_v(u, \mu^\star) + c\, pb_v(\mu^\star, u) \\ &= (a+b+c)\, pb_v(\mu^\star, u) \end{aligned}$$

$[1-(a+b+c)]pb_v(\mu^\star, u) \leq 0.$

But, $a + b + c < 1$

$\Rightarrow 1-(a+b+c) > 0.$

Hence, $pb_v(\mu^\star, u) = 0$

$\Rightarrow \mu^\star = u,$

which proves the uniqueness.

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