Some Common Fixed Point Theorems for Hardy Roger Type Contraction on Partial b-Metric Spaces

Dr. A. Mary Priya Dharsini[1] , Dr.A. Leema Maria Prakasam[2], Dr. A. Jennie Sebasty Pritha[3]
1,2,3 - Assistant Professor,
PG and Research Department of Mathematics,
Holy Cross College (Autonomous),
Tiruchirappalli – 620002.
Mail id : priyairudayam@gmail.com

Abstract
We prove some fixed point theorems using Hardy Roger type contraction in the setting of b-metric as well as partial b-metric spaces in order to find the existence and uniqueness of the common fixed point. We also provide examples to illustrate the existence of fixed point and its uniqueness.
Keywords: Common fixed point, Hardy roger type contraction, Partial b-metric
Mathematics Subject Classification: Primary 47H10; Secondary 54H25

1. Introduction

Fixed point theory is the most important and unique instrument in the field of Science, Engineering and Technological World. The method of fixed point theory is used in analysis from 20th Century onwards. It was introduced by Joseph Liouville in 1837 and by Charles Emile Picard in 1890 based on the method of successive approximations and it is relevant in finding the existence of solutions in differential equations.

The pioneering work of Classical Theory was given by Stephan Banach which was established in 1922. In point of the historical view, there are some Mathematicians who completed the results in Fixed Point Theory, they are L.E.T. Brower, W.A. Kirk, Silms, Granas and Dugundiji.

The concept of b – metric space was introduced by Bakhtin in 1989. Further it was worked out and expanded by Czerwik in 1993. Making use of their results as better tools, many scholars derived some renowned Banach fixed point theorems in the b - metric spaces and partial b-metric spaces. The partial b-metric was introduced by O'Neill and it is also known as dualistic partial metric space.

2. Preliminaries

Definition 2.1: [11]

Let χ_b be a non-empty set and $q_b \geq 1$ be a given real number and also the mapping $d_{b_m}: \chi_b \times \chi_b \to R^+$ (R^+stands for non-negative real) when it satisfies following conditions ,

for all $g_b, i_b, k_b \in \chi_b$,

- $d_{b_m}(g_b, i_b) = 0 \; if \; and \; only \; if \; g_b = i_b$,
- $d_{b_m}(g_b, i_b) = (i_b, g_b)$,
- $d_{b_m}(g_b, k_b) \leq \; q_b\left[d_{b_m}(g_b, i_b) + (i_b, k_b)\right]$

The pair $\left(\chi_b, d_{b_m}\right)$ is called b- metric space.

It is the extension of usual metric space.

Definition 2.2: [8]

Let Φ_s be the non-empty set and $p_m: \Phi_s \times \Phi_s \to [0, \infty)$ be a mapping, then the following conditions are satisfied, for all $\tau, \varphi, \gamma \in \Phi_s$:

- $\tau = \varphi \Leftrightarrow p_m(\tau, \tau) = p_m(\varphi, \varphi) = p_m(\tau, \varphi)$;
- $p_m(\tau, \tau) \leq p_m(\tau, \varphi)$;
- $p_m(\tau, \varphi) = p_m(\varphi, \tau)$;
- $p_m(\tau, \varphi) \leq p_m(\tau, \gamma) + p_m(\gamma, \varphi) - p_m(\gamma, \gamma)$

In which the pair (Φ_s, p_m) is called as partial metric space.

Definition 2.3: [12]

Let χ_b be a non-empty set and $q_b \geq 1$ be a given real number and $\left(\chi_b, p_{b_m}\right)$ be a partial b-metric space when it fulfills the following conditions for all $g_b, i_b, k_b \in \chi_b$,

- $g_b = i_b$ if and only if $p_{b_m}(g_b, g_b) = p_{b_m}(g_b, i_b) = p_{b_m}(i_b, i_b)$;
- $p_{b_m}(g_b, g_b) \leq p_{b_m}(g_b, i_b)$;
- $p_{b_m}(g_b, i_b) = p_{b_m}(i_b, g_b)$;
- $p_{b_m}(g_b, i_b) \leq \; q_b\left[p_{b_m}(g_b, k_b) + p_{b_m}\left(k_{b,} i_b\right) - p_{b_m}\left(k_{b,} k_b\right)\right]$

This is known as partial b- metric space.

The number $q_b \geq 1 \; is \; the \; coefficient \; of \; \left(\chi_b, p_{b_m}\right)$.

Definition 2.4: [12]

Let χ_b be a non-empty set and $q_b \geq 1$ be a given real number and $\left(\chi_b, p_{b_m}, \; q_b\right)$ be a partial b-metric space. g_{b_n} be any sequence in χ_b, and $g_b \in \chi_b, then,$

- The sequence $\left\{ g_{b_n}\right\}$ is said to be convergent if it converges to $g_b \; if \; \lim_{n\to\infty} p_{b_m}\left(g_{b_n}, g_b\right)$ exists and is finite.
- The $\left\{ g_{b_n}\right\}$ seqeuence is said to be Cauchy sequence in $\left(\chi_b, p_{b_m}, \; q_b\right)$ if $\lim_{n,m\to\infty} d_{b_m}\left(g_{b_n}, g_{b_m}\right)$ exists and finite.

- (χ_b, p_{b_m}, q_b) is called as complete partial b-metric space if for all Cauchy Sequence g_{b_n} $in\ \chi_b\ and\ there\ exists$ $g_b \in \chi_b\ such\ that,$
$\lim_{n,m\to\infty} d_{b_m}(g_{b_n}, g_{b_m}) = \lim_{n\to\infty} p_{b_m}(g_{b_n}, g_b) = p_{b_m}(g_b, g_b).$

We should remember that limit of the convergent sequence may not be unique.

Definition 2.5: [5]

Given a metric space $(\Phi_{s,} d_b)$, the mapping $T_{bm}: \Phi_s \to \Phi_s$ is said to be an interpolative Kannan contraction mapping if,

$$d_b(T_{b_m}\zeta_m, T_{b_m}\chi_m) \le \eta_s(d_b(\zeta_m, T_{b_m}))^{\tau'} \cdot d_b(\chi_m, T_{b_m}\zeta_m)^{1-\tau'},$$

Then it is called Interpolative Kannan type Contracrion.

Definition 2.6: [7]

Let $(\Phi_{s,} d_b)$ be a complete metric space. We can say that the self – mapping $T_{b_m}: \Phi_s \to \Phi_s$ is an $interpolative\ Hardy\ Rogers\ type\ contraction$ if there exists $\eta_s \in [0,1)$ and $\tau', \sigma', \rho' \in (0,1)$ with $\tau' + \sigma' + \rho' < 1, such\ that,$

$$d_b(T_{b_m}\zeta_m, T_{b_m}\chi_m) \le \eta_s \left[(d_b(\zeta_m, \chi_m))^{\sigma'} . (d_b(\zeta_m, T_{b_m}\zeta_m))^{\tau'} . (d_b(\chi_m, T_{b_m}\chi_m))^{\rho'}\right] \cdot (1/2\,[d_b(\zeta_m, T_{b_m}\chi_m) + d_b(\chi_m, T_{b_m}\zeta_m)])^{1-\tau'-\sigma'-\rho'}, \quad (1)$$

For all $\zeta_m, \chi_m \in \phi_s \backslash Fix(T_{b_m})$.

Main Results

3. Fixed Point theorems with the Contraction on b-Metric Spaces and Partial b-Metric Spaces

We begin the chapter by giving the notion of

$Interpolative\ Hardy - Rogers\ type\ contractions.$

Definition: 3.1

Let $(\Phi_{s,} d_b)$ be a metric space. We can say that the self – mapping $T_{bm}: \Phi_s \to \Phi_s$ is an $interpolative\ Hardy\ Rogers\ type\ contraction$ if there exists $\eta_s \in [0,1)$ and $\tau', \sigma', \rho' \in (0,1)$ with $\tau' + \sigma' + \rho' < 1, such\ that,$

$$d_b(T_{b_m}\zeta_m, T_{b_m}\chi_m) \le \eta_s \left[(d_b(\zeta_m, \chi_m))^{\sigma'} . (d_b(\zeta_m, T_{b_m}\zeta_m))^{\tau'} . (d_b(\chi_m, T_{b_m}\chi_m))^{\rho'}\right] \cdot (1/2\,[d_b(\zeta_m, T_{b_m}\chi_m) + d_b(\chi_m, T_{b_m}\zeta_m)])^{1-\tau'-\sigma'-\rho'} \quad (2)$$

For all $\zeta_m, \chi_m \in \phi_s \backslash Fix(T_{b_m})$.

Theorem 3.2

Let $(\Phi_{s,} d_b)$ a complete b-metric space, $q_b \ge 1$ be a real number and T_{b_m} be an interpolative Hardy Rogers type contraction. Then T_{b_m}has a unique fixed point in Φ_s.

Proof:

Given,

Let $(\Phi_{s,}d_b)$ be a complete b-metric space and T_{b_m} be an interpolative type contraction.

To prove,

T_{b_m} has a fixed point in ϕ_s .

As we start from $\zeta_{m_0} \in \phi_s$, considering $\{\zeta_{m_n}\}$, given as

$\zeta_{m_n} = {T_{b_m}}^n(\zeta_{m_0})$ for each positive integer n.

If there exists n_o such that $\zeta_{m_{n_0}} = \zeta_{m_{n_0}+1}$, then $\zeta_{m_{n_0}}$ is a fixed point of T_{b_m}.

Let us assume that $\zeta_{m_n} \neq \zeta_{m_n+1}$ for all $n \geq 0$.

By substituting the values $\zeta_m = \zeta_{m_n}$ and $\chi_m = \zeta_{m_{n-1}}$ in the notion of interpolative Hardy- Rogers type contraction,

$$d_b(T_{b_m}\zeta_m, T_{b_m}\chi_m) \leq \eta_s \left[(d_b(\zeta_m, \chi_m))^{\sigma'} . (d_b(\zeta_m, T_{b_m}))^{\tau'} . (d_b(\chi_m, T_{b_m}\chi_m))^{\rho'} . \right]$$

$$\cdot (1/2 \left[d_b(\zeta_m, T_{b_m}\chi_m) + d_b(\chi_m, T_{b_m}\zeta_m) \right])^{1-\tau'-\sigma'-\rho'} \quad (3)$$

We shall consider $\zeta_m = \zeta_{m_n}$, $\chi_m = \zeta_{m_n-1}$, and we get ,

$$d_b(\zeta_{m_n+1}, \zeta_{m_n})$$
$$= d_b(T_{b_m}\zeta_{m_n}, T_{b_m}\zeta_{m_n-1})$$
$$\leq \eta_s[d_b(\zeta_{m_n}, \zeta_{m_n-1})]^{\sigma'} \cdot [d_b(\zeta_{m_n}, T_{b_m}\zeta_{m_n})]^{\tau'} \cdot [d_b(\zeta_{m_n-1}, T_{b_m}\zeta_{m_n-1})]^{\rho'} \cdot [1/2 (d_b(\zeta_{m_n}, T_{b_m}\zeta_{m_n-1}) + d_b(\zeta_{m_n-1}, T_{b_m}\zeta_{m_n}))]^{1-\tau'-\sigma'-\rho'}$$

$$\leq \eta_s[d_b(\zeta_{m_n}, \zeta_{m_n-1})]^{\sigma'} \cdot [d_b(\zeta_{m_n}, \zeta_{m_n+1})]^{\tau'} \cdot [d_b(\zeta_{m_n-1}, \zeta_{m_n})]^{\rho'} \cdot [1/2 (d_b(\zeta_{m_n}, \zeta_{m_n}) + d_b(\zeta_{m_n-1}, \zeta_{m_n+1}))]^{1-\tau'-\sigma'-\rho'}$$

Since, $d_b(\zeta_{m_n}, \zeta_{m_n}) = 0$, we have,

$$\leq \eta_s[d_b(\zeta_{m_n}, \zeta_{m_n-1})]^{\sigma'} \cdot [d_b(\zeta_{m_n}, \zeta_{m_n+1})]^{\tau'} \cdot [d_b(\zeta_{m_n-1}, \zeta_{m_n})]^{\rho'} \cdot [1/2 (d_b(\zeta_{m_n-1}, \zeta_{m_n+1}))]^{1-\tau'-\sigma'-\rho'}$$

$$\leq \eta_s[d_b(\zeta_{m_n}, \zeta_{m_n-1})]^{\sigma'} \cdot [d_b(\zeta_{m_n}, \zeta_{m_n+1})]^{\tau'} \cdot [d_b(\zeta_{m_n-1}, \zeta_{m_n})]^{\rho'} \cdot [1/2 (d_b(\zeta_{m_n-1}, \zeta_{m_n}) + d_b(\zeta_{m_n}, \zeta_{m_n+1}))]^{1-\tau'-\sigma'-\rho'} \quad (3)$$

Supposing that,

$d_b(\zeta_{m_n-1}, \zeta_{m_n}) < d_b(\zeta_{m_n}, \zeta_{m_n+1})$ for some $n \geq 1$, thus we get the result as,

$$1/2 (d_b(\zeta_{m_n-1}, \zeta_{m_n}) + d_b(\zeta_{m_n}, \zeta_{m_n+1})) \leq d_b(\zeta_{m_n}, \zeta_{m_n+1})$$

So, the inequality (2) can be written as,

$$\leq \eta_s \left[d_b(\zeta_{m_n}, \zeta_{m_n-1})\right]^{\sigma'} \cdot \left[d_b(\zeta_{m_n}, \zeta_{m_n+1})\right]^{\tau'} \cdot \left[d_b(\zeta_{m_n-1}, \zeta_{m_n})\right]^{\rho'}$$
$$\cdot \left[{}^1/_2 \left(d_b(\zeta_{m_n-1}, \zeta_{m_n}) + d_b(\zeta_{m_n}, \zeta_{m_n+1})\right)\right]$$
$$\cdot \left[{}^1/_2 \left(d_b(\zeta_{m_n-1}, \zeta_{m_n}) + d_b(\zeta_{m_n}, \zeta_{m_n+1})\right)\right]^{1-(\tau'+\sigma'+\rho')}$$

$$d_b(\zeta_{m_n}, \zeta_{m_n+1})$$
$$\leq \eta_s\left[d_b(\zeta_{m_n}, \zeta_{m_n-1})\right]^{\sigma'} \cdot \left[d_b(\zeta_{m_n}, \zeta_{m_n+1})\right]^{\tau'} \cdot \left[d_b(\zeta_{m_n-1}, \zeta_{m_n})\right]^{\rho'}$$
$$\cdot \left[d_b(\zeta_{m_n}, \zeta_{m_n+1}))\right] \cdot \left[d_b(\zeta_{m_n}, \zeta_{m_n+1}))\right]^{-\tau'-\sigma'-\rho'}$$

$$\left[d_b(\zeta_{m_n}, \zeta_{m_n+1}))\right]^{\sigma'+\rho'} \leq \eta\left[d_b(\zeta_{m_n}, \zeta_{m_n-1})\right]^{\sigma'+\rho'}, \forall\, n \geq 1 \qquad (4)$$

Now we can conclude that,

$\left[d_b(,\zeta_{m_n-1}, \zeta_{m_n})\right] \geq \left[d_b(\zeta_{m_n}, \zeta_{m_n+1}))\right]$, This is contradiction to the assumption that we had considered already.

Hence we derive as the conclusion that,

$$\left[d_b(\zeta_{m_n}, \zeta_{m_n+1}))\right] \leq \left[d_b(\zeta_{m_n-1}, \zeta_{m_n})\right], \quad \forall\, n \geq 1.$$

Thus,

$\left[d_b(\zeta_{m_n-1}, \zeta_{m_n})\right]$ is a non – increasing sequence with positive terms.

The set we have,

$$\varphi = \lim_{n\to\infty} d_b(\zeta_{m_n-1,}\zeta_{m_n})$$

So, we get,

$$\left[{}^1/_2 \left(d_b(\zeta_{m_n-1}, \zeta_{m_n}) + d_b(\zeta_{m_n}, \zeta_{m_n+1})\right)\right] \leq d_b(\zeta_{m_n-1}, \zeta_{m_n}), \forall\, n \geq 1$$

So, the inequality (2) can be changed as

$$d_b(\zeta_{m_n}, \zeta_{m_n+1}) \leq \eta_s\left[d_b(\zeta_{m_n}, \zeta_{m_n-1})\right]^{\sigma'} \cdot \left[d_b(\zeta_{m_n}, \zeta_{m_n+1})\right]^{\tau'} \cdot$$
$$\left[d_b(\zeta_{m_n-1}, \zeta_{m_n})\right]^{\rho'} \cdot \left[d_b(\zeta_{m_n-1}, \zeta_{m_n})\right] \cdot$$
$$\left[d_b(\zeta_{m_n}, \zeta_{m_n-1}))\right]^{-\tau'-\sigma'-\rho'}$$

$$d_b(\zeta_{m_n}, \zeta_{m_n+1})^{1-\tau'} \leq \eta_s \left[d_b(\zeta_{m_n-1}, \zeta_{m_n})\right]^{1-\tau'} \quad ,\forall\, n \geq 1 \qquad (5)$$

Now we shall reduce the equation as follows. Hence we derive it as,

$$d_b(\zeta_{m_n}, \zeta_{m_n+1}) \leq \eta_s\left[d_b(\zeta_{m_n-1}, \zeta_{m_n})\right]$$
$$\leq {\eta_s}^2\left[d_b(\zeta_{m_n-2}, \zeta_{m_n-1})\right]$$
$$\leq {\eta_s}^3\left[d_b(\zeta_{m_n-3}, \zeta_{m_n-2})\right]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$d_b(\zeta_{m_n}, \zeta_{m_n+1}) \leq {\eta_s}^n\left[d_b(\zeta_{m_n-n}, \zeta_{m_n-(n+1)})\right]$$
$$d_b(\zeta_{m_n}, \zeta_{m_n+1}) \leq {\eta_s}^n\left[d_b(\zeta_{m_0}, \zeta_{m_1})\right]$$

From the assumption we derived as $n < 1$ by taking $n \to \infty$ the inequality (4),

We shall derive that $\varphi = 0$,

It follows that we can prove that $\{\zeta_{m_n}\}$ is a Cauchy sequence by deriving with the standard tools.

Starting with the triangle inequality we receive the following estimation, making use of the following standard results based on the inequality of complete b – metric space.

Let $q_b \geq 1$ be the coefficient. Considering in the following equations we get,

$$d_b(\zeta_{m_n}, \zeta_{m_n+r}) \leq q_b[(d_b(\zeta_{m_n}, \zeta_{m_n+1}) + d_b(\zeta_{m_n+1}, \zeta_{m_n+r}))]$$

$$\leq q_b\left[(d_b(\zeta_{m_n}, \zeta_{m_n+1}) + q_b\begin{bmatrix} d_b(\zeta_{m_n+1}, \zeta_{m_n+2}) \\ +q_b[d_b(\zeta_{m_n+2}, \zeta_{m_n+r}))] \end{bmatrix}\right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\leq q_b d_b(\zeta_{m_n}, \zeta_{m_n+1}) + q^2{}_b d_b(\zeta_{m_n+1}, \zeta_{m_n+2}) + q^3{}_b d_b(\zeta_{m_n+2}, \zeta_{m_n+3}) + \ .\ldots + q^r{}_b d_b(\zeta_{m_n+r-1}, \zeta_{m_n+r})$$

$$\leq q_b d_b(\zeta_{m_n}, \zeta_{m_n+1}) + q^2{}_b d_b(\zeta_{m_n+1}, \zeta_{m_n+2}) + q^3{}_b d_b(\zeta_{m_n+2}, \zeta_{m_n+3}) + \ .\ldots + q^r{}_b d_b(\zeta_{m_n+r-1}, \zeta_{m_n+r})$$

$$\leq q_b \eta_s{}^n d_b(\zeta_{m_0}, \zeta_{m_1}) + q^2 \eta_s{}^{n+1}{}_b d_b(\zeta_{m_0}, \zeta_{m_1}) + q^3 \eta_s{}^{n+2}{}_b d_b(\zeta_{m_0}, \zeta_{m_1}) + \ .\ldots + q^r{}_b \eta_s{}^{n+r-1} d_b(\zeta_{m_0}, \zeta_{m_1})$$

$$\leq q_b \eta_s{}^n d_b(\zeta_{m_0}, \zeta_{m_1})\{1 + q_b\eta_s + q_b{}^2\eta_s{}^2 + \cdots + q_b{}^{r-1}\eta_s{}^{r-1}\}$$

$$\leq q_b \eta_s{}^n d_b(\zeta_{m_0}, \zeta_{m_1})\ \{1 - q_\eta\}^{-1}$$

(Since $(1-x)^{-1} = 1 + x + x^2 \ldots\ldots$))

$$\leq q_b \eta_s{}^n d_b(\zeta_{m_0}, \zeta_{m_1})\left(\frac{1}{1 - q_b\eta_s}\right)$$

$$d_b(\zeta_{m_n}, \zeta_{m_n+r}) \leq \left(\frac{q_b\eta_s{}^n}{1-q_b\eta_s}\right) d_b(\zeta_{m_0}, \zeta_{m_1}) \qquad (7)$$

So, $\{\zeta_{m_n}\}$ is a Cauchy sequence in the complete b-metric space (φ_s, d_b) and so, there exists $\zeta_m \in \varphi_s$ such that $\lim_{n\to\infty} d_b(\zeta_{m_n}, \zeta_m) = 0$

Supposing that $\zeta_m \neq T_{b_m}\zeta_m$ for each $n \geq 0$, by letting $\zeta_m = \zeta_{m_n}$ and χ_m in (1)

Thus we get,

$$d_b(\zeta_{m_{n+1}}, T_{b_m}\zeta_{m_n}) = d_b(T_{b_m}\zeta_{m_n}, T_{b_m}\zeta_m)$$

$$\leq \eta_s[d_b(\zeta_{m_n}, \zeta_m)]^{\sigma'} \cdot [d_b(\zeta_{m_n}, T_{b_m}\zeta_{m_n})]^{\tau'} \cdot [d_b(\zeta_m, T_{b_m}\zeta_m)]^{\rho'} \cdot \left[\left(d_b(\zeta_m, T_{b_m}\zeta_m)\right)\right] + \left[\left(d_b(\zeta_m, T_{b_m}\zeta_{m_n})\right)\right]^{-\tau'-\sigma'-\rho'} \qquad (8)$$

Let $n \to \infty$, in the inequality (7) we conclude that $\left(d_b(\zeta_m, T_{b_m}\zeta_m)\right) = 0$ which contradicts our assumption.

Thus, $T_{b_m}\zeta_m = \zeta_m$

Now let us prove its uniqueness.

If $\zeta_m{}'$ be any fixed point of T_{b_m} such that and apply it in equation (8),

We consider $d_b(\zeta_m, \zeta_m{}') = d_b(T_{b_m}\zeta_m, T_{b_m}\zeta_m{}')$

$\leq \eta_s\left[d_b(\zeta_m, \zeta_m{}')\right]^{\sigma'} \cdot \left[d_b(\zeta_m, T_{b_m}\zeta_m)\right]^{\tau'} \cdot$

$\left[d_b(\zeta_m{}', T_{b_m}\zeta_m{}')\right]^{\rho'} \cdot \left[\left(d_b(\zeta_m{}', T_{b_m}\zeta_m{}')\right)\right] + \left[\left(d_b(\zeta_m{}', T_{b_m}\zeta_m)\right)\right]^{-\tau'-\sigma'-\rho'}$

As $n \to \infty$ in this inequality, $d_b(\zeta_m{}', T_{b_m}\zeta_m) = 0$,

It's a contradiction so we get,

$$T_{b_m}\zeta_m = \zeta_m{}'$$

Therefore, $\zeta_m{}' = \zeta_m$

Hence proved.

4. We can now derive the analog of the main theorem in the setting of Complete Partial b-Metric Spaces.

As we begin we shall have the lemma which is used in the following theorem.

Lemma 4.1:

Let p_b be a complete partial b-metric on a non-empty set $\Phi_{s,}$ and φ_{p_b} be the corresponding standard metric space on the same set $\Phi_{s,}$ then,

- A sequence $\{\Psi_{p_n}\}$ is the fundamental in the framework of a partial b-metric $(\Phi_{s,}p_b)$, if and only if it is a fundamental sequence in the setting of the corresponding standard metric space $(\Phi_{s,}\varphi_{p_b})$
- A partial b-metric space$(\Phi_{s,}p_b)$ is complete if and only if the corresponding standard metric space $(\Phi_{s,}\varphi_{p_b})$ is complete.

  Moreover,

  $$\lim_{n\to\infty} \varphi_{p_b}(\Psi_p, \Psi_{p_n}) = 0 \Leftrightarrow p_b(\Psi_p, \Psi_p) = \lim_{n\to\infty} p_b(\Psi_p, \Psi_{p_n}) = \lim_{n,m\to\infty} p_b(\Psi_{p_n}, \Psi_{p_m})$$

- If $\Psi_{p_n} \to \tau$ as $n \to \infty$ in a partial b-metric space $(\Phi_{s,}p_b)$ with
- $p_b(\tau, \tau) = 0$, then we have,

  $$\lim_{n\to\infty} p_b(\Psi_{p_n}, \rho_p) = p_b(\tau, \rho_p) \text{ for every } \rho_p \in \Phi_s.'$$

So according to this Lemma, the sequence $\{\Psi_{p_n}\}$ is the fundamental sequence in the standard metric $(\Phi_{s,}\varphi_{p_b})$.

Because, $(\Phi_{s,}p_b)$ is complete, $(\Phi_{s,}\varphi_{p_b})$ is also complete.

Theorem: 4.2

Let $(\Phi_{s,}p_b)$ be a complete partial b-metric space with coefficient $s_b \geq 1$.

Let, $T_{bm}: \Phi_s \to \Phi_s$ be a given mapping. Suppose there exists $\eta \in [0,1)$ and $\tau', \sigma', \rho' \in (0,1)$ with $\tau' + \sigma' + \rho' < 1, such\ that,$

$$p_b(T_{b_m}\Psi_p, T_{b_m}\vartheta_p) \leq \eta\left[\left(p_b(\Psi_p, \vartheta_p)\right)^{\sigma'} . (p_b(\Psi_p, T_{b_m}\vartheta_p))^{\tau'} . (p_b(\vartheta_p, T_{b_m}\vartheta_p))^{\rho'}\right] \cdot \left(\tfrac{1}{2}\left[p_b(\Psi_p, T_{b_m}\vartheta_p) + p_b(\vartheta_p, T_{b_m}\Psi_p)\right]\right)^{1-\tau'-\sigma'-\rho'} \quad (12)$$

For all $\Psi_p, \vartheta_p \in \phi_s \backslash Fix(T_{b_m})$.

Proof:

For any $\Psi_{p_0} \in (\Phi_s, p_b)$, we form a sequence $\{\Psi_{p_n}\}$ by $\Psi_{p_n} = T^n{}_{bm}(\Psi_{p_0})$ for each n$\in \mathbb{N}$.

If there exists n_0 such that $\Psi_{p_{n_0}} = \Psi_{p_{n_{0+1}}}$, then $\Psi_{p_{n_0}}$ is a fixed point of T_{b_m}.

So the proof gets completed.

So, now we shall assume that, $\Psi_{p_n} = \Psi_{p_{n+1}}$ for each n ≥ 0.

Through submitting the values of $\Psi_p = \Psi_{p_n}$ and $\vartheta_p = \Psi_{p_{n-1}}$ in (12),

So we get as ,

$$p_b(\Psi_{p_{n+1}}, \Psi_{p_n})$$
$$= p_b(T_{b_m}\Psi_{p_n}, T_{b_m}\Psi_{p_{n-1}})$$
$$\leq \eta\, [p_b(\Psi_{p_n}, \Psi_{p_{n-1}})]^{\sigma'} \cdot [p_b(\Psi_{p_n}, T_{b_m}\Psi_{p_n})]^{\tau'} \cdot [p_b(\Psi_{p_{n-1}}, T_{b_m}\Psi_{p_n-1})]^{\rho'} \cdot [{}^1/_2\,(p_b(\Psi_{p_n}, T_{b_m}\Psi_{p_n-1}) + p_b(\Psi_{p_n-1}, T_{b_m}\Psi_{p_n}))]^{1-\tau'-\sigma'-\rho'}$$
$$\leq \eta\, [p_b(\Psi_{p_n}, \Psi_{p_n-1})]^{\sigma'} \cdot [p_b(\Psi_{p_n}, \Psi_{p_{n+1}})]^{\tau'} \cdot [p_b(\Psi_{p_n-1}, \Psi_{p_n})]^{\rho'} \cdot [{}^1/_2\,(p_b(\Psi_{p_n}, \Psi_{p_n}) + p_b(\Psi_{p_n-1}, \Psi_{p_{n+1}}))]^{1-\tau'-\sigma'-\rho'}$$

Since, $p_b(\Psi_{p_n}, \Psi_{p_n}) = 0$, we have,

$$\leq \eta\, [p_b(\Psi_{p_n}, \Psi_{p_n-1})]^{\sigma'} \cdot [p_b(\Psi_{p_n}, \Psi_{p_{n+1}})]^{\tau'} \cdot [p_b(\Psi_{p_n-1}, \Psi_{p_n})]^{\rho'} \cdot [{}^1/_2\,(p_b(\Psi_{p_n-1}, \Psi_{p_{n+1}}))]^{1-\tau'-\sigma'-\rho'}$$
$$\leq \eta\, [p_b(\Psi_{p_n}, \Psi_{p_n-1})]^{\sigma'} \cdot [p_b(\Psi_{p_n}, \Psi_{p_{n+1}})]^{\tau'} \cdot [p_b(\Psi_{p_n-1}, \Psi_{p_n})]^{\rho'} \cdot [{}^1/_2\,(p_b(\Psi_{p_n}, \Psi_{p_n-1}) + p_b(\Psi_{p_{n+1}}, \Psi_{p_n}))]^{1-\tau'-\sigma'-\rho'} \qquad (13)$$

Supposing that,

$[p_b(\Psi_{p_n-1}, \Psi_{p_n})] \leq [p_b(\Psi_{p_n}, \Psi_{p_{n+1}})]$,

So the inequality (13) becomes as,

$[p_b(\Psi_{p_n}, \Psi_{p_{n+1}})] \leq \eta\, p_b(\Psi_{p_n}, \Psi_{p_{n+1}})$

This is contradiction since we have considered that $\eta < 1$.

Hence we derive as the conclusion that,

$[p_b(\Psi_{p_n}, \Psi_{p_{n+1}}))] \leq [p_b(\Psi_{p_n-1}, \Psi_{p_n})], \quad \forall\, n \geq 1$.

Thus,

$[p_b(\Psi_{p_n-1}, \Psi_{p_n})]$ is a non – increasing sequence with positive terms.

The set we have,

$[{}^1/_2\,(p_b(\Psi_{p_n-1}, \Psi_{p_n}) + p_b(\Psi_{p_n}, \Psi_{p_{n+1}}))] \leq p_b(\Psi_{p_n-1}, \Psi_{p_n}), \forall\, n \geq 1$

So, the inequality (12) can be changed as

$$p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) \leq \eta\, [p_b(\Psi_{p_n}, \Psi_{p_n-1})]^{\sigma'} \cdot [p_b(\Psi_{p_n}, \Psi_{p_{n+1}})]^{\tau'} \cdot [p_b(\Psi_{p_n-1}, \Psi_{p_n})]^{\rho'} \cdot [p_b(\Psi_{p_n-1}, \Psi_{p_n})] \cdot \quad [p_b(\Psi_{p_n}, \Psi_{p_n-1}))]^{-\tau'-\sigma'-\rho'}$$

$$p_b(\Psi_{p_n}, \Psi_{p_{n+1}})^{1-\tau'} \leq \eta\, [p_b(\Psi_{p_n-1}, \Psi_{p_n})]^{1-\tau'} \quad ,\forall\, n \geq 1 \qquad (14)$$

So , there exists a non negative constant m such that,

$\lim_{n\to\infty} p_b(\Psi_{p_n-1}, \Psi_{p_n}) = m$. Here $m \geq 0$.

So, we get in the inequality (14) as,

$$p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) \leq \eta^{\gamma}[p_b(\Psi_{p_n-1}, \Psi_{p_n})]$$
$$\leq \eta^{2\gamma}[p_b(\Psi_{p_n-2}, \Psi_{p_n-1})]$$
$$\leq \eta^{3\gamma}[p_b(\Psi_{p_n-3}, \Psi_{p_n-2})]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) \leq \eta^{n\gamma}[p_b(\Psi_{p_n-n}, \Psi_{p_n-(n+1)})]$$
$$p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) \leq \eta^{n\gamma}[p_b(\Psi_{p_0}, \Psi_{p_1})] \qquad (15)$$

Since η, $\gamma < 1$we have $\tilde{\eta} = \eta^{\gamma} < 1$.

Thus letting $n \to \infty$ in (15), we derive that $m = 0$

It follows that, we can prove that $\{\Psi_{p_n}\}$ is a fundamental Cauchy sequence by deriving with the standard tools.

Starting with the triangle inequality of the partial b-metric spaces, we receive the following estimation, making use of the following standard results based on the inequality of complete partial b – metric space.

Let $q_b \geq 1$ be the coefficient. Considering in the following equations we get,

$$p_b(\Psi_{p_n}, \Psi_{p_{n+r}}) \leq s_b[(p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) + p(\Psi_{p_{n+1}}, \Psi_{p_{n+r}}))]$$

$$\leq s_b\left[(p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) + s_b\left[p_b(\Psi_{p_{n+1}}, \Psi_{p_{n+2}}) + s_b[p_b(\Psi_{p_{n+2}}, \Psi_{p_{n+r}}))]\right]\right]$$

$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$

$$\leq s_b p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) + {s_b}^2 p_b(\Psi_{p_{n+1}}, \Psi_{p_{n+2}}) \quad + {s_b}^3 p_b(\Psi_{p_{n+2}}, \Psi_{p_{n+3}}) + \cdots + {s_b}^r p_b(\Psi_{p_{n+r-1}}, \Psi_{p_{n+r}})$$

$$\leq s_b p_b(\Psi_{p_n}, \Psi_{p_{n+1}}) + {s_b}^2 p_b(\Psi_{p_{n+1}}, \Psi_{p_{n+2}}) + {s_b}^3 p_b(\Psi_{p_{n+2}}, \Psi_{p_{n+3}}) \quad + \cdots . + {s_b}^r p_b(\Psi_{p_{n+r-1}}, \Psi_{p_{n+r}})$$

$$\leq s_b\tilde{\eta}^n p_b(\Psi_{p_0},\Psi_{p_1}) + s_b{}^2\tilde{\eta}^{n+1}{}_b p_b(\Psi_{p_0},\Psi_{p_1}) + s_b{}^3\tilde{\eta}^{n+2}{}_b p_b(\Psi_{p_0},\Psi_{p_1}) + \cdots + s_b{}^r\tilde{\eta}^{n+r-1} p_b(\Psi_{p_0},\Psi_{p_1})$$

$$\leq s_b\tilde{\eta}^n p_b(\Psi_{p_0},\Psi_{p_1})\{1 + s_b\tilde{\eta} + s_b{}^2\tilde{\eta}^2 + \cdots + s_b{}^{r-1}\tilde{\eta}^{r-1}\}$$

$$\leq s_b\tilde{\eta}^n p_b(\Psi_{p_0},\Psi_{p_1})\left\{1 - s_{b_\eta}\right\}^{-1}$$

(Since $(1-x)^{-1} = 1 + x + x^2 \ldots\ldots$)

$$\leq s_b\tilde{\eta}^n p_b(\Psi_{p_0},\Psi_{p_1})\left(\frac{1}{1 - s_b\tilde{\eta}}\right)$$

$$p_b(\Psi_{p_n},\Psi_{p_n+r}) \leq \left(\frac{s_b\tilde{\eta}^n}{1-s_b\tilde{\eta}}\right) p_b(\Psi_{p_0},\Psi_{p_1}) \quad (17)$$

The sequence $\{\Psi_{p_n}\}$ is the fundamental sequence in $(\Phi_{s,}p_b)$, as $n \to \infty$

So according to the Lemma, the sequence $\{\Psi_{p_n}\}$ is the fundamental sequence in the standard metric $(\Phi_{s,}\varphi_{p_b})$.

Because, $(\Phi_{s,}p_b)$ is complete, $(\Phi_{s,}\varphi_{p_b})$ is also complete.

So there exists, $\Psi_p \in \Phi_s$ such that,

$$p_b(\Psi_p, \Psi_p) = \lim_{n\to\infty} p_b(\Psi_p, \Psi_{p_n}) = \lim_{n,m\to\infty} p_b(\Psi_{p_n}, \Psi_{p_m}) = 0 \quad (18)$$

Thus we get,

$$\lim_{n\to\infty} \varphi_{p_b}(\Psi_p, \Psi_{p_n}) = 0 \quad (19)$$

Now we shall show that the limit Ψ_p of the iterative sequence $\{\Psi_{p_n}\}$ is a fixed point of the mapping T_{b_m}.

Taking it as the assumption is that we get,

$$\Psi_p \neq T_{b_m}\Psi_p \quad (20)$$

So that we derive it as,

$$p_b(\Psi_p, T_{b_m}\Psi_p) > 0$$

We know that, $\Psi_{p_n} \neq T_{b_m}\Psi_p$ for each $n \geq 0, as\ we\ let\ \ \Psi_p = \Psi_{p_n}$ in (12) we get that,

$$p_b(\Psi_{p+1}, T_{b_m}\Psi_p) = p_b(T_{b_m}\Psi_{p_n}, T_{b_m}\Psi_p)$$

$$\leq \eta_s\left[p_b(\Psi_{p_n},\Psi_p)\right]^{\sigma'} \cdot \left[p_b(\Psi_{p_n},T_{b_m}\Psi_{p_n})\right]^{\tau'} \cdot \left[p_b(\Psi_p,T_{b_m}\Psi_p)\right]^{\rho'} \cdot \left[\frac{1}{2}\left[p_b(\Psi_{p_n},T_{b_m}\Psi_p) + p_b(\Psi_p,T_{b_m}\Psi_{p_n})\right]\right]^{1-\tau'-\sigma'-\rho'}$$

$$= \eta_s\left[p_b(\Psi_{p_n},\Psi_{p_n})\right]^{\sigma'} \cdot \left[p_b(\Psi_{p_n},\Psi_{p_{n+1}})\right]^{\tau'} \cdot \left[p_b(\Psi_p,T_{b_m}\Psi_p)\right]^{\rho'} \cdot \left[\frac{1}{2}\left[p_b(\Psi_{p_n},T_{b_m}\Psi_p) + p_b(\Psi_p,T_{b_m}\Psi_{p_n})\right]\right]^{1-\tau'-\sigma'-\rho'} \quad (21)$$

As $n \to \infty$ in the inequality (21) we can find that ,

$$p_b(\Psi_p, T_{b_m}\Psi_p) = 0$$

So, $\Psi_p = T_{b_m}\Psi_p$

ISSN 2515-8260 Volume 07, Issue 09, 2020

Which contradicts our assumption, so, we find that,

$T_{b_m}\Psi_p = \Psi_p$.

Hence the theorem.

Refrences:

[1] Kumara. S , Topology of Metric Spaces.

[2] Pawan K.Jain and Khail Ahamed, Metric spaces.

[3] Krein.S.G, Petunin, Interpolation of Linear operators; American Mathematical Society Providence RI, USA, 1978.

[4] Karapınar, E. Revisiting the Kannan type contractions via interpolation, Theory non linear Analysis.Appl., 2018.2, 85-87.

[5] Karapınar.E, Agarwal, Interpolative contraction on partial b-metric spaces, Mathematics 2018, 6-256.

[6] Banach, S. Sur les operations dans les esembles abstraites estt luer applications , Fund Math, 1922. 3, 133-138.

[7] Hardy, G.E, Rogers, T.D. A generalization of a fixed point theorem. 1973, 201-206

[8] Mathews S. G, Partial Metric Topology, 1994.

[9] Ayidi, Karapınar, Agarwal, Generalized contraction on b-metric and partial b-metric spaces, 2012

[10] Mlaiki, N. Abodayeh, K.Ayidi, Related Fixed point theorems ,2018.

[11] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, In: Functional Analysis, Vol. 30, pp 26-37, Ul'yasnovsk Gos.Ped.Inst, Ul'yanovsk (1989)

[12] Shukla.S, "Partial b-metric spaces and fixed point theorems", Mediterr. J. Math. vol.11 (2014), 703-711.