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A STUDY OF FIXED POINT THEOREMS IN S-METRIC SPACES

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Abstract

In this paper, we define a new class of generalized contractive condition for proving some common fixed point results for four mappings satisfying in complete S-metric spaces. Our results generalize and unify some results in the recent literature and illustrate the unique common fixed point of four self-maps through some example.

Keywords: fixed point, contractive, compatible, continuous, metric space

1. Introduction and Preliminaries

Banach's contraction principle is most important results in theory of fixed point. With this concept many mathematicians introduced various metric spaces like b-metric, G-metric, D-metric etc., In addition to that Sedghi[1] introduced the S-metric space and some fixed point theorems on it. He also introduced the $\boldsymbol{D}^*$ metric space from the D-metric space. Currently fixed point theory attracts many researchers to discover various results on it. S-metric space is further extended by K. Prudhvi, Animesh Gupta and so on. Many authours [2,3,4,5] define compatible mappings to prove the unique common fixed point in S-metric spaces.

Definition 1.1:

Let $\boldsymbol{\mathcal{J}}$ be a nonempty set. An S-metric space is a function

$S: \mathcal{J}^3 \to [0, \infty]$ that satisfies the following conditions for all $r, s, t, b \in \mathcal{J}$

(1) $S(r, s, t) = 0$ if and only if $r = s = t$.

(2) $S(r, s, t) \leq S(r, r, b) + S(s, s, b) + S(t, t, b)$

The pair $(\boldsymbol{\mathcal{J}}, \mathbf{S})$is called on S-metric space.

(1) Let $\mathbb{R}$ be a real line. Then

$$S(r, s, t) = |r + t - 2s| + |r - t|,$$

for all $r, s, t, b \in \mathbb{R}$ is a S-metric on $\mathbb{R}$.

$$S(r, r, r) = |r + r - 2r| + |r - r| = 0$$

$$S(r, s, t) = |r + t - 2s| + |r - t|$$

$$S(r,s,t) \leq |r+b-2r| + |r-b| + |s+b-2s| + |s-b|$$
$$+|t-b-2t| + |t-b|$$

$S(r,s,t) \leq S(r,r,b) + S(s,s,b) + S(t,t,b)$

Take $r = 2, s = 3, t = 6, b = 4$

$S(2,3,6) = |2+6-2(3)| + |2-6| = 6$

$S(2,2,4) = |2+4-2(2)| + |2-4| = 4$

$S(3,3,4) = |3+4-2(3)| + |3-4| = 2$

$S(6,6,4) = |6+4-2(6)| + |6-4| = 4$

$S(2,3,6) \leq S(2,2,4) + S(3,3,4) + S(6,6,4)$

$$6 \leq 4+2+4$$
$$6 \leq 10$$

$S(r,s,t)$ is a S-metric space.

Definition 1.2[1]:

Let ($\mathcal{J}$, S) be an S-metric space and $G \subset \mathcal{J}$. A sequence $\{u_n\}$ in $\mathcal{J}$converges to u if $S(u_n, u_{n,} u) \to 0$ as $n \to \infty$, that is for every $\varepsilon > 0$ there exists $n_0 \in N$ such that for $n \geq n_0$, $S(u_n, u_{n,} u) < \varepsilon$. We denote $\lim_{n\to\infty} u_n = u$ and we say that u is the limit of $\{u_n\}$ in $\mathcal{J}$.

Definition 1.3[1]:

Let ($\mathcal{J}$, S) be an S-metric space and $G \subset \mathcal{J}$. A sequence $\{u_n\}$ in $\mathcal{J}$ is said to beCauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in N$ such that $S(u_n, u_{n,} u) < \varepsilon$, for each $n, m \geq n_0$.

Definition 1.4[2]:

Let ($\mathcal{J}$, S) and $(\mathcal{J}', S')$ be two S-metric space, and $l: (\mathcal{J}, S) \longrightarrow (\mathcal{J}', S')$ be a function. Then l is said to be continuous at a point $b \in \mathcal{J}$ if and only if for every sequence u_n in $\mathcal{J}$, $S(u_n, u_{n,} b) \to 0$ implies $S'(l(u_n), l(u_n), l(b)) \to 0$. A function l is continuous at $\mathcal{J}$ if and only if it is continuous at all $b \in \mathcal{J}$.

Definition 1.5:

Let ($\mathcal{J}$, S) be an S-metric space. A pair $\{l, k\}$ is said to becompatible if only if $\lim_{n\to\infty} S(lku_n, lku_n, lku_n) = 0$, whenever $\{u_n\}$ is a sequence in $\mathcal{J}$ such that $\lim_{n\to\infty} lu_n = \lim_{n\to\infty} ku_n = r$ for some $r \in \mathcal{J}$.

Lemma 1.6:

In an S-metric space, we have $S(u,u,v) = S(v,v,u)$.

Lemma 1.7[3]:

Let ($\mathcal{J}$, S) be an S-metric space. If there exist sequences $\{u_n\}$ and $\{v_n\}$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$, then $\lim_{n\to\infty} S(u_n, u_n, v_n) = S(u,u,v)$.

Lemma 1.8[2]:

Let ($\mathcal{J}$, S) be an S-metric space. If there exist two sequences $\{u_n\}$ and $\{v_n\}$

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such that $\lim_{n\to\infty} S(u_n, u_n, v_n) = 0$, whenever $\{u_n\}$ is a sequence in χ such that $\lim_{n\to\infty} u_n = r$ for some $r \in$ χ, then$\lim_{n\to\infty} v_n = r$.

2. MAIN RESULTS

Theorem 2.1:

In a complete S-metric space $(\mathcal{J}, S)$, the self-maps arel, k, A and B with $l(\mathcal{J}) \subseteq B(\mathcal{J}), k(\mathcal{J}) \subseteq A(\mathcal{J})$ and $\{l, A\}$ and $\{k, B\}$ are compatible. If

$$S(lu, lv, kw) \leq p \max\{S(lu, lu, kw), S(lv, lv, Au), S(Au, Av, Bw), S(kw, kw, Bw)\} \tag{2.1}$$

For each $, v, w \in \mathcal{J}$, $0 < p < 1$.

Also A and B are continuous.

Then l, k, A and B have a unique common fixed point in $\mathcal{J}$.

Proof:

Let $u_0 \in \mathcal{J}$.

Since $l(\mathcal{J}) \subseteq B(\mathcal{J})$, there exists $u_1 \in \mathcal{J}$ such that $lu_0 = Bu_1$.

Also, $ku_1 \in A(\mathcal{J})$, we choose $u_2 \in \mathcal{J}$ such that $ku_1 = Au_2$.

In general, $u_{2n+1} \in \mathcal{J}$ is chosen such that $lu_{2n} = Bu_{2n+1}$and $u_{2n+2} \in \mathcal{J}$ such that $ku_{2n+1} = Au_{2n+2}$.

$\{v_n\} \in \mathcal{J}$ is obtained such that

$$v_{2n} = lu_{2n} = Bu_{2n+1}$$

$$v_{2n+1} = ku_{2n+1} = Au_{2n+2}, \qquad n \geq 0$$

To prove:$\{v_n\}$ is a Cauchy sequence.

$$\begin{aligned} S(v_{2n}, v_{2n}, v_{2n+1}) &= S(lu_{2n}, lu_{2n}, ku_{2n+1}) \\ &\leq p \max\{S(lu_{2n}, lu_{2n}, ku_{2n+1}), S(lu_{2n}, lu_{2n}, Au_{2n}), \\ &\qquad S(Au_{2n}, Au_{2n}, Bu_{2n+1}), S(ku_{2n+1}, ku_{2n+1}, Bu_{2n+1})\} \\ &= p \max\{S(v_{2n}, v_{2n}, v_{2n+1}), S(v_{2n}, v_{2n}, v_{2n-1}), \\ &\qquad S(v_{2n-1}, v_{2n-1}, v_{2n}), S(v_{2n+1}, v_{2n+1}, v_{2n})\} \end{aligned}$$

$$= p\, max\{S(v_{2n-1}, v_{2n-1}, v_{2n}), S(v_{2n}, v_{2n}, v_{2n+1})\} \tag{2.2}$$

If $S(v_{2n}, v_{2n}, v_{2n+1}) > S(v_{2n-1}, v_{2n-1}, v_{2n})$, then by (2.2)

$$S(v_{2n}, v_{2n}, v_{2n+1}) < p\, S(v_{2n}, v_{2n}, v_{2n+1})$$

which is a contradiction.

Hence $S(v_{2n}, v_{2n}, v_{2n+1}) \leq S(v_{2n-1}, v_{2n-1}, v_{2n})$

Therefore by (2.2)

$$S(v_{2n}, v_{2n}, v_{2n+1}) \leq p\, S(v_{2n-1}, v_{2n-1}, v_{2n}) \tag{2.3}$$

$$\begin{aligned} S(v_{2n-1}, v_{2n-1}, v_{2n}) &= S(v_{2n}, v_{2n}, v_{2n-1}) \\ &= S(lu_{2n}, lu_{2n}, ku_{2n-1}) \\ &\leq p \max\{S(lu_{2n}, lu_{2n}, ku_{2n-1}), S(lu_{2n}, lu_{2n}, Au_{2n}), \\ &\qquad S(Au_{2n}, Au_{2n}, Bu_{2n-1}), S(ku_{2n-1}, ku_{2n-1}, Bu_{2n-1})\} \\ &= p \max\{S(v_{2n}, v_{2n}, v_{2n-1}), S(v_{2n}, v_{2n}, v_{2n-1}), \end{aligned}$$

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$S(v_{2n-1}, v_{2n-1}, v_{2n-2}), S(v_{2n-1}, v_{2n-1}, v_{2n-2})\}$
$= p\, max\{S(v_{2n-2}, v_{2n-2}, v_{2n-1}), S(v_{2n}, v_{2n}, v_{2n-1})\}$ (2.4)
If $S(v_{2n}, v_{2n}, v_{2n-1}) > S(v_{2n-2}, v_{2n-2}, v_{2n-1})$, then by (4)
$S(v_{2n}, v_{2n}, v_{2n-1}) < p\, S(v_{2n}, v_{2n}, v_{2n-1})$
which is a contradiction.
Hence $S(v_{2n-1}, v_{2n-1}, v_{2n}) \leq S(v_{2n-2}, v_{2n-2}, v_{2n-1})$
Therefore, by (4)
$S(v_{2n-1}, v_{2n-1}, v_{2n}) \leq p\, S(v_{2n-2}, v_{2n-2}, v_{2n-1})$ (2.5)
From (2.3) and (2.5)
$S(v_n, v_n, v_{n-1}) \leq p\, S(v_{n-1}, v_{n-1}, v_{n-2}), \quad n \geq 2$
where $0 < p < 1$.
Hence for $n \geq 2$, it follows that
$S(v_n, v_n, v_{n-1}) \leq \cdots \leq p^{n-1} S(v_1, v_1, v_0)$ (2.6)
For $n > m$, by triangle inequality in S-metric space we have
$S(v_n, v_n, v_m) \leq 2S(v_m, v_m, v_{m+1}) + 2S(v_{m+1}, v_{m+1}, v_{m+2}) + \cdots$
$+2S(v_{n-1}, v_{n-1}, v_n)$
From (2.6) we have
$S(v_n, v_n, v_m) \leq 2(p^m + p^{m+1} + \cdots + p^{n-1})\, S(v_1, v_1, v_0)$
$\leq 2\, p^m(1 + p + p^2 + \cdots)\, S(v_1, v_1, v_0)$
$\leq 2\frac{p^m}{1-q} S(v_1, v_1, v_0) \to 0$ as $m \to 0, 0 < p < 1$.
Therefore $\{v_n\}$ is a Cauchy sequence.
Since $\mathcal{J}$ is a complete S-metric space for some $v \in \mathcal{J}$ such that

$$\lim_{n\to\infty} lu_{2n} = \lim_{n\to\infty} Bu_{2n+1} = \lim_{n\to\infty} ku_{2n+1} = \lim_{n\to\infty} Au_{2n+2} = v$$

Claim:v is a common fixed point of l, k, A and B.
Since A is continuous,

$$\lim_{n\to\infty} A^2u_{2n+2} = Av, \quad \lim_{n\to\infty} Alu_{2n} = Av$$

Since l and A are compatible,

$$\lim_{n\to\infty} S(lAu_{2n}, lAu_{2n}, Alu_{2n}) = 0$$

By lemma 1.8,

$$\lim_{n\to\infty} lAu_{2n} = Av$$

Put $u = v = Au_{2n}$ and $w = u_{2n+1}$ in (1)

$S(lAu_{2n}, lAu_{2n}, ku_{2n+1}) \leq$ p $\max\{S(lAu_{2n}, lAu_{2n}, ku_{2n+1}),$
$S(lAu_{2n}, lAu_{2n}, A^2u_{2n}),$
$S(A^2u_{2n}, A^2u_{2n}, Bu_{2n+1}),$
$S(ku_{2n+1}, ku_{2n+1}, Bu_{2n+1})\}$ (2.7)

In (2.7), apply upper limit $\boldsymbol{n} \to \infty$
$S(Av, Av, v) = \lim_{n\to\infty} S(lAu_{2n}, lAu_{2n}, ku_{2n+1})$
$\leq$ p $\max\{\lim_{n\to\infty} S(lAu_{2n}, lAu_{2n}, ku_{2n+1}),$

$$\lim_{n\to\infty} S(lAu_{2n}, lAu_{2n}, A^2u_{2n}), \lim_{n\to\infty} S(A^2u_{2n}, A^2u_{2n}, Bu_{2n+1}),$$
$$\lim_{n\to\infty} S(ku_{2n+1}, ku_{2n+1}, Bu_{2n+1})\}$$

$S(Av, Av, v) \leq p \max\{S(Av, Av, v), S(Av, Av, Av), S(Av, Av, v), S(v, v, v)\}$
$= p\, S(Av, Av, v)$

$S(Av, Av, v) \leq p\, S(Av, Av, v)$, $0 < p < 1$ it follows that $Av = v$.

Since B is continuous,

$$\lim_{n\to\infty} B^2u_{2n+1} = Bv, \quad \lim_{n\to\infty} Bku_{2n+1} = Bv$$

Since k and B are compatible,

$$\lim_{n\to\infty} S(kBu_{2n+1}, kBu_{2n+1}, Bku_{2n+1}) = 0$$

By lemma 1.8,

$$\lim_{n\to\infty} kBu_{2n+1} = Bv$$

Put $u = v = u_{2n}$ and $z = Bu_{2n+1}$ in (2.1)

$S(lu_{2n}, lu_{2n}, kBu_{2n+1}) \leq p \max\{S(lu_{2n}, lu_{2n}, kBu_{2n+1}),$
$S(lu_{2n}, lu_{2n}, Au_{2n}),$
$S(Au_{2n}, Au_{2n}, B^2u_{2n+1}),$
$S(kBu_{2n+1}, kBu_{2n+1}, B^2u_{2n+1})\}$ (2.8)

In (2.8), apply upper limit when $n \to \infty$

$S(v, v, Bv) = \lim_{n\to\infty} S(lu_{2n}, lu_{2n}, kBu_{2n+1})$
$\leq p \max\{\lim_{n\to\infty} S(lu_{2n}, lu_{2n}, kBu_{2n+1}),$
$\lim_{n\to\infty} S(lu_{2n}, lu_{2n}, Au_{2n}),$
$\lim_{n\to\infty} S(Au_{2n}, Au_{2n}, B^2u_{2n+1}),$
$\lim_{n\to\infty} S(kBu_{2n+1}, kBu_{2n+1}, B^2u_{2n+1})\}$
$\leq p \max\{S(v, v, Bv), S(v, v, v), S(v, v, Bv), S(Bv, Bv, Bv)\}$
$\leq p\{ S(v, v, Bv)\}$

$S(v, v, Bv) \leq p\{ S(v, v, Bv)\}$, $0 < p < 1$ it follows that $Bv = v$.

Also, we can apply (1)

$S(lv, lv, ku_{2n+1}) \leq p \max\{S(lv, lv, ku_{2n+1}), S(lv, lv, Av),$
$S(Av, Av, Bu_{2n+1}), S(ku_{2n+1}, ku_{2n+1}, Bu_{2n+1})\}$ (2.9)

In (2.9), apply upper limit $n \to \infty$ as $Av = Bv = v$

$S(lv, lv, v) \leq p \max\{S(lv, lv, v), S(lv, lv, v), S(Av, Av, v), S(v, v, v)\}$
$= pS(lv, lv, v)$

Since $0 < p < 1$, $S(lv, lv, v) = 0$ and $lv = v$

By using (2.1) and $Av = Bv = lv = v$

$S(v, v, kv) = S(lv, lv, kv)$
$\leq p \max\{S(lv, lv, kv), S(lv, lv, Av), S(Av, Av, Bv), S(kv, kv, Bv)\}$
$= p \max\{S(v, v, kv), S(v, v, v), S(v, v, v), S(kv, kv, v)\}$
$= p\, S(v, v, kv)$

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$S(v, v, kv) = 0$ and $kv = v$

$Av = Bv = lv = kv = v$ is proved

If there exist another common fixed point $u \in \mathcal{J}$ of all l, k, A and B, then

$S(u, u, v) = S(lu, lu, kv)$

$$\leq p \max\{S(lu, lu, kv), S(lu, lu, Au), S(Au, Au, Bv), S(kv, kv, Bv)\}$$

$$= p \max\{S(u, u, v), S(u, u, u), S(u, u, v), S(v, v, v)\}$$

$$= p\, S(u, u, v)$$

which implies that $S(u, u, v) = 0$ and $u = v$

Therefore, v is a unique common fixed point of l, k, A and B.

Hence proved.

The following example explains the theorem (2.1)

Example 2.2:

Let $\chi = [0,1]$ be S-metric with $S(u, v, w) = |u - w| + |v - w|$. Define l, k, A and B on $\mathcal{J}$ by

$$l(u) = \left(\frac{u}{3}\right)^8, k(u) = \left(\frac{u}{3}\right)^4, \qquad A(u) = \left(\frac{u}{3}\right)^2, B(u) = \frac{u}{3}$$

Obviously, $l(\mathcal{J}) \subseteq B(\mathcal{J})$ and $k(\mathcal{J}) \subseteq A(\mathcal{J})$.

Furthermore, the pairs $\{l, A\}$ and $\{k, B\}$ are compatible mappings.

Solution:

Also for each $u, v, w \in \mathcal{J}$, we have

$$S(lu, lv, kw) = |lu - kw| + |lv - kw|$$

$$= \left|\left(\frac{u}{3}\right)^8 - \left(\frac{w}{3}\right)^4\right| + \left|\left(\frac{v}{3}\right)^8 - \left(\frac{w}{3}\right)^4\right|$$

$$= \left|\left(\frac{u}{3}\right)^4 + \left(\frac{w}{3}\right)^2\right|\left|\left(\frac{u}{3}\right)^4 - \left(\frac{w}{3}\right)^2\right| + \left|\left(\frac{v}{3}\right)^4 + \left(\frac{w}{3}\right)^2\right| \left|\left(\frac{v}{3}\right)^4 - \left(\frac{w}{3}\right)^2\right|$$

$$= \left|\left(\frac{u}{3}\right)^4 + \left(\frac{w}{3}\right)^2\right|\left|\left(\frac{u}{3}\right)^2 + \frac{w}{3}\right|\left|\left(\frac{u}{3}\right)^2 - \frac{w}{3}\right| + \left|\left(\frac{v}{3}\right)^4 + \left(\frac{w}{3}\right)^2\right| \left|\left(\frac{v}{3}\right)^2 + \frac{w}{3}\right|\left|\left(\frac{v}{3}\right)^2 - \frac{w}{3}\right|$$

$$= \frac{10}{81}\left|\left(\frac{u}{3}\right)^2 + \frac{w}{3}\right|\left|\left(\frac{u}{3}\right)^2 - \frac{w}{3}\right| + \frac{10}{81}\left|\left(\frac{v}{3}\right)^2 + \frac{w}{3}\right|\left|\left(\frac{v}{3}\right)^2 - \frac{w}{3}\right|$$

$$= \frac{10}{81} \times \frac{4}{9}\left|\left(\frac{u}{3}\right)^2 - \frac{w}{3}\right| + \frac{10}{81} \times \frac{4}{9}\left|\left(\frac{v}{3}\right)^2 - \frac{w}{3}\right|$$

$$= \frac{40}{729}\left|\left(\frac{u}{3}\right)^2 - \frac{w}{3}\right| + \frac{40}{729}\left|\left(\frac{v}{3}\right)^2 - \frac{w}{3}\right|$$

$$= \frac{40}{729}|Au - Bw| + \frac{40}{729}|Av - Bw|$$

$$= \frac{40}{729}\, S(Au, Av, Bw)$$

$$\leq \frac{40}{729} \max \begin{Bmatrix} S(lu, lv, kw), S(lv, lv, Au), S(Au, Av, Bw) \\ S(kw, kw, Bw) \end{Bmatrix}$$

where $40/_{729} \leq p < 1$.

Thus l, k, A and B satisfy the conditions given in theorem 2.1.And 0 is the unique common fixed point of l, k, A and B.

CONCLUSION

In this paper, we proved the common fixed point theorem for the pairs of compatible mappings. With the help of complete S-metric space the common fixed point theorem were found and the unique common fixed point of four self-maps were also identified. Numerical examples were also illustrated.

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