On Paranormed Difference C_2-Sequence Spaces In 2-Normes Spaces

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Abstract. In this paper, paranormed C_2-sequence spaces are defined in the 2-normed space. The topological properties of these sequence space are also studied. The solidness and the symmetry of the sequence space are also verified for the ℓ and ℓ_∞ base spaces.
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Introduction

In past times, a plenty of works have been done on various kinds of paranormed spaces. This concept of paranorm is related to linear metric space and the study of this on special type of spaces i.e. Sequence spaces was initiated by Maddox (1969) and then it is carried forward by many more researchers.

Before proceeding towards main results, we will firstly recall some of the useful notations and basic definitions.

Definition 1.1: A Vector space X having zero element α along with a function $\mathrm{P}: S \to R^+$ (called a paranorm on S) then (X, P) is called a paranormed space if P satisfies the following conditions:

(i) $\mathrm{P}(\alpha) = 0$

(ii) $\mathrm{P}(s) = \mathrm{P}(-s)$

(iii) $\mathrm{P}(s_1 + s_2) \le \mathrm{P}(s_1) + \mathrm{P}(s_2)$

(iv) Scalar multiplication is continous i.e., if $\{\gamma_n\} \to \gamma$ as n$\to \infty$ and $\{s_n\}$ is a sequence of vectors with $\mathrm{P}(s_n - s) \to 0$ as n$\to \infty$, then $\mathrm{P}(\gamma_n\, s_n - \gamma s) \to 0$ as $n \to \infty$.

If the function P satisfies an additional property i.e. if $\mathrm{P}(s) = 0 \Rightarrow s = \alpha$, then the paranorm is called total paranorm. (cf. Wilansky (1978)).

The study on paranormed sequence spaces was done by Feyzi Basar and Medine Yesilkayagil(2019) [1-4] and many others and also studied various types of paranormed sequence spaces and function spaces.

Definition 1.2: Let X be a linear space with $Dim(X)_K > 1$ over the field $\mathbb{R}$(or $\mathbb{C}$). A 2-Norm on space X is real valued function $\|.,.\|$ on $X \times X$ satisfying the following conditions:

(i) $\| x, y \| \ge 0$ and $\| x, y \| = 0$ iff x and y are linearly dependent

(ii) $\| x, y \| = \| y, x \| \ \forall\, x, y \in X$

(iii) $\| \beta x, y \| = |\beta| \, \| x, y \|$, where $\beta \in K$ and $x, y \in X$

(iv) $\| x + y, z \| \le \| x, z \| + \| y, z \|$, for all $x, y, z \in X$

Then the pair $(X, \|.,.\|)$ is called a 2-normed space.

A sequence $\{a_n\}$ in a linear 2-normed space X is said to be Cauchy sequence if $\lim_{m,n\to\infty} \| a_n - a_m, t \| = 0 \ \forall\, t \in X$.

The concept of 2-normed space was studied by Iseki (1976)[8], it is a generalization of normed linear space.

Definition 1.3: The notation used to denote the set of bicomplex numbers is C_2 and the sets $\mathbb{R}$ and $\mathbb{C}$ are denoted by C_0 , C_1, respectively . the set of bicomplex no. is defined as

$$C_2 = \{ a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 : a_k \in C_0, k = 1,2,3,4\} = \{ w_1 + i_2 w_2 : w_1, w_2 \in C_1\}$$

where $i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1$.

There exist only two non-trivial idempotent elements in the set of bicomplex numbers C_2 denoted by I_1 and I_2 given as $I_1 = \frac{1+i_1 i_2}{2}$ and $I_2 = \frac{1-i_1 i_2}{2}$. Note that $I_1 + I_2 = 1$ and $I_1 . I_2 = 1$. The number $\xi = w_1 + i_2 w_2$ can be uniquely expressed as a complex combination of $I_1$and I_2.

$$\xi = w_1 + i_2 w_2 = {}^1\xi I_1 + {}^2\xi I_2$$

where ${}^1\xi = w_1 - i_1 w_2$ and ${}^2\xi = w_1 + i_1 w_2$. The coefficients involved in the representation of ξ which are ${}^1\xi$ and ${}^2\xi$ are called idempotent components of ξ and ${}^1\xi I_1 + {}^2\xi I_2$ is known as idempotent representation of bicomplex number ξ.

The collection of components of the elements of space is denoted by A_1 and A_2 defined as follows:

$A_1 = \{ {}^1\xi : \xi \in C_2 \}$ and $A_2 = \{ {}^2\xi : \xi \in C_2\}$

Both the spaces are known as auxiliary complex spaces.

The norm in C_2 is defined as follows:

$$\|\xi\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} = \sqrt{|w_1|^2 + |w_2|^2} = \sqrt{\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}} \quad (1.1)$$

The inequality defined for the norm of product of two numbers of the space is given by :

$$\|\xi \times \eta\| \leq \sqrt{2} \, \| \xi \| \, \| \eta \|$$

The inequality given above is best possible relation. For this reason we call C_2 as modified complex Banach algebra.

Throughout the paper, the ω_4, c_0, c denote the space of all the bicomplex sequences, null sequences, convergent sequences and $t_k = \{s_k^{-1}\}$. (For details cf. Mursaleen [5]).

Definition 1.4: The function $\mathcal{M}$ defined as $\mathcal{M}: [0, \infty) \to [0, \infty)$ is said to be orlicz function if it satisfies the following conditions:

(a) $\mathcal{M}$ is continuous.
(b) $\mathcal{M}$ is non-decreasing function satisfying $\mathcal{M}(0) = 0$, $\mathcal{M}(x) > 0$ for $x > 0$.
(c) $\mathcal{M}(\lambda x + (1-\lambda)y) \leq \lambda \mathcal{M}(x) + (1-\lambda)\mathcal{M}(y)$ for $\lambda \in (0,1)$;

and if the condition of convexity is replaced by $\mathcal{M}(x+y) \leq \mathcal{M}(x) + \mathcal{M}(y)$; then the Orlicz function $\mathcal{M}$ is called Modulus function.

Definition 1.5: Let us now introduce a new concept of difference sequence spaces

$$\mathcal{B}(\Delta_m^n) = \{x = (x_k) \in \omega : (\Delta_m^n x_k) \in \mathcal{B}\}$$

where $\Delta_m^n x = \Delta_m^n (x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_k)$ and $\Delta_m^0 x_k = x_k$, $\forall \, k \in \mathbb{N}$, which is given by binomial expansion as:

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}$$

For more details of difference sequence space refer to book by M.Et. and Colak [4].

2. Paranormed C_2-Sequence Spaces

In this section we give some result on the linearity, completeness of the paranormed difference C_2 sequence space.

Let us define a double norm on C_2 space:

$$\|\xi,\eta\| = \| {}^1\xi e_1 + {}^2\xi e_2 , {}^1\eta e_1 + {}^2\eta e_2 \| = | {}^1\xi . {}^2\eta - {}^2\xi . {}^1\eta |$$

By using Orlicz function $\mathcal{M}$, we define sequence spaces on the bicomplex space as follows:

$$\ell(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M}) = \left\{ \{\xi_k\} \in \omega_4 : \sum_{k=1}^{\infty} \left[\mathcal{M}\left(\frac{\|\Delta_m^n(\xi_k), t\|}{K}\right)\right]^{s_k} < \infty, K > 0, \forall\, t \in C_2 \right\}$$

Lemma 2.1: The function $\|.,.\|$ defined as:

$$\| \xi_k, t \|_{\mathcal{M}} = \inf\left\{ K > 0 : \sum_{k=1}^{\infty} \left[\mathcal{M}\left(\frac{\|\Delta_m^n(\xi_k), t\|}{K}\right)\right]^{s_k} \le 1;\ \forall\, t \in C_2 \right\}$$

is a norm on $\ell(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$.

Theorem 2.1. The space $\ell(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ is a banach space in the norm $\|.,.\|_{\mathcal{M}}$.

Remark 2.1: We know that $(C_2, \|.,.\|)$ is a normed space by the norm $\|.,.\|$ defined in (1.1). Let $\mathcal{M}$ be an Orlicz functions. We define the following Orlicz C_2-sequence spaces for all $t \in C_2$:

$$\ell(C_2, \|.,.\|,\ \Delta_m^n, s, \mathcal{M}) = \left\{ \{\xi_k\} \in \omega_4 : \sum_{k=1}^{\infty} \left[\mathcal{M}\left(\frac{\|\Delta_m^n(\xi_k), t\|}{K}\right)\right]^{s_k} < \infty, K > 0 \right\}$$

$$c(C_2, \|.,.\|,\ \Delta_m^n, s, \mathcal{M}) = \left\{ \{\xi_k\} \in \omega_4 : \sum_{k=1}^{\infty} \left[\mathcal{M}\left(\frac{\|\Delta_m^n(\xi_k) - L, t\|}{K}\right)\right]^{s_k} \to 0, L \in C_2, K > 0 \right\}$$

$$c_0(C_2, \|.,.\|,\ \Delta_m^n, s, \mathcal{M}) = \left\{ \{\xi_k\} \in \omega_4 : \sum_{k=1}^{\infty} \left[\mathcal{M}\left(\frac{\|\Delta_m^n(\xi_k), t\|}{K}\right)\right]^{s_k} \to 0, K > 0 \right\}$$

$$\ell^{\infty}(C_2, \|.,.\|,\ \Delta_m^n, s, \mathcal{M}) = \left\{ \{\xi_k\} \in \omega_4 : \sup_k \left[\mathcal{M}\left(\frac{\|\Delta_m^n(\xi_k), t\|}{K}\right)\right]^{s_k} < \infty, K > 0 \right\}$$

Proposition 2.1. Any C_2- sequence $\{\eta_k\}$ belongs to $X(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ if and only if $\{{}^1\eta_k\} \in S(A_1, \|.,.\|,\ \Delta_m^n, s, \mathcal{M})$ and $\{{}^2\eta_k\} \in S(A_2, \|.,.\|,\ \Delta_m^n, s, \mathcal{M})$, where $S = \ell, c, c_0$ and ℓ^{∞}.

Theorem 2.2. The set $\ell^{\infty}(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ is linear space over C_1 for any sequence $s = \{s_k\} \in R^+$.

Proof : Let $\{\xi_k\}$ and $\{\eta_k\}$ be the two sequences of the space $\ell^{\infty}(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ .then let us suppose $\exists\, K_1 > 0, K_2 > 0$ such that

$$\sup_k \left[\mathcal{M}\left(\frac{\|\Delta_m^n(\xi_k), t\|}{K_1}\right)\right]^{s_k} < \infty \; and \; \sup_k \left[\mathcal{M}\left(\frac{\|\Delta_m^n(\eta_k), t\|}{K_2}\right)\right]^{s_k} < \infty$$

Let us consider $\beta, \gamma \in C_1, K = \max\{2|\beta|K_1, 2|\gamma|K_2\}$. So that

$$\sup_k \left[\mathcal{M}\left(\frac{\|\Delta_m^n(\beta\xi_k + \gamma\eta_k), t\|}{K}\right)\right]^{s_k}$$

$$= \sup_k \left[\mathcal{M}\left(\frac{\|(\beta\,\Delta_m^n\,\xi_k + \gamma\,\Delta_m^n\,\eta_k), t\|}{K}\right)\right]^{s_k}$$

$$\le K \sup_k \left[\mathcal{M}\left(\frac{\|\Delta_m^n(\xi_k), t\|}{K_1}\right)\right]^{s_k} + K \sup_k \left[\mathcal{M}\left(\frac{\|\Delta_m^n(\eta_k), t\|}{K_2}\right)\right]^{s_k} < \infty$$

Therefore $\{\beta\xi_k + \gamma\eta_k\} \in \ell^{\infty}(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$. Hence, $\ell^{\infty}(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ is a linear space over C_1.

Theorem 2.3. For every sequence $s = \{s_k\}$ of strictly positive real numbers, the sets $\ell(C_2, \|.,.\|,\ \Delta_m^n, s, M)$, $c(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ and $c_0(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ are linear spaces over C_1.

Theorem 2.4. The space $\ell^{\infty}(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ is paranormed by

$$p(\eta) = \| \eta_1, t \| + \inf\left\{K^{s_k/p} : \sup\left\{\mathcal{M}\left(\frac{\| \Delta_m^n (\eta_k), t \|}{K}\right)(x_k)^{\frac{1}{s_k}}\right\} \leq 1, K > 0\right\}$$

where $P = \max\{1, \sup s_k\}$.

Proof: For the null C_2 -sequence θ, $q(\theta_1) = 0$ *and* $\mathcal{M}\left(\frac{\|\Delta_m^n(\theta_k), t\|}{K}\right) = 0$ for any $K > 0$.

Therefore, $p(\theta) = 0$.

Also, $p(-\eta) = p(\eta), \forall \eta \in l^{\infty}(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$.

Now, let $\eta, \xi \in \ell^{\infty}(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$. Then there exist $K_1, K_2 > 0$ such that

$$\mathcal{M}\left(\frac{\| \Delta_m^n (\eta_k), t \|}{K_1}\right)(x_k)^{\frac{1}{s_k}} \leq 1 \text{ and } \mathcal{M}\left(\frac{\| \Delta_m^n (\xi_k), t \|}{K_2}\right)(x_k)^{\frac{1}{s_k}} \leq 1$$

Suppose $K = K_1 + K_2$. Then

$$\sup_k \left\{\mathcal{M}\left(\frac{\| \Delta_m^n (\eta_k + \xi_k), t \|}{K_1}\right)(x_k)^{\frac{1}{s_k}}\right\}$$

$$\leq \frac{K_1}{K} \sup_k \left\{\mathcal{M}\left(\frac{\| \Delta_m^n (\eta_k), t \|}{K}\right)(x_k)^{\frac{1}{s_k}}\right\} + \frac{K_2}{K} \sup_k \left\{\mathcal{M}\left(\frac{\| \Delta_m^n (\xi_k), t \|}{K}\right)(x_k)^{\frac{1}{s_k}}\right\}$$

$$\leq 1$$

Also,

$p(\xi + \eta) =$

$$\| \xi_1 + \eta_1, t \| + \inf\{K^{\frac{s_k}{p}} : \sup_k \left[\mathcal{M}\left(\frac{\| \Delta_m^n (\xi_k + \eta_k), t \|}{K}\right)(x_k)^{\frac{1}{s_k}}\right] \quad \leq 1, K > 0, \forall t \in C_2 \}$$

$$\leq \| \eta_1, t \| + \inf\left\{K_1^{\frac{s_k}{P}} : \sup\left\{\mathcal{M}\left(\frac{\|\Delta_m^n(\eta_k), t\|}{K_1}\right)(x_k)^{\frac{1}{s_k}}\right\} \leq 1 \right\}$$

$$+ \| \xi_1, t \| + \inf\left\{K_2^{\frac{s_k}{P}} : \sup\left\{\mathcal{M}\left(\frac{\|\Delta_m^n(\xi_k), t\|}{K_2}\right)(x_k)^{\frac{1}{s_k}}\right\} \leq 1 \right\} \leq p(\xi) + p(\eta)$$

Let $\alpha \in C_0$. Then

$$p(\alpha\eta) = \|\alpha\eta_1, t\| + \inf\{K^{\frac{s_k}{p}} : \sup\left\{\mathcal{M}\left(\frac{\| \Delta_m^n (\eta_k), t \|}{K}\right)(x_k)^{\frac{1}{s_k}}\right\} \leq 1, K > 0, \forall t \in C_2\}$$

$$\leq \sqrt{2}\|\alpha\|\|\eta_1, t\| + \inf\{(\|\alpha\|H)^{\frac{s_k}{p}} : \sup_k \left\{\mathcal{M}\left(\frac{\| \Delta_m^n (\eta_k), t \|}{K}\right)(x_k)^{\frac{1}{s_k}}\right\} \leq 1, H > 0\}$$

where $H = K \backslash \| \alpha \|$.

Theorem 2.5. $\ell^{\infty}(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ is a complete paranormed space for $s \in \ell^{\infty}(C_0)$.

Proof. Let $\{\eta_k^m\}$ be a Cauchy sequence in $\ell^{\infty}(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$. Then for all $n \in N$, $p(\eta_k^i - \eta_k^j) \to 0$ as $i, j \to \infty$.

Let for the given $\epsilon > 0$, there exist a $H > 0$ and some $x > 0$ such that $\frac{\epsilon}{xH} > 0$ and $sup_n (s_n)^{t_n} \leq \mathcal{M}\left(\frac{xH}{2}\right)$.

Now for $p(\eta_k^i - \eta_k^j) \to 0$ as $i, j \to \infty$, there exists some $k_0 \in N$ such that

$$p(\eta_k^i - \eta_k^j) < \frac{\epsilon}{xH}, \forall i, j \geq k_0, \forall n \in \mathbb{N}.$$

Therefore,

$$\left\|\sum_{r=1}^{m-1} \eta_r^i - \sum_{r=1}^{m-1} \eta_r^j, t\right\| + \inf\left\{K^{\frac{s_n}{p}} : \sup\left\{\mathcal{M}\left(\frac{\|\Delta_m^n \eta_k^i - \Delta_m^n \eta_k^j, t\|}{K}\right)(x_k)^{\frac{1}{s_k}}\right\} \leq 1, K > 0,\right\} < \frac{\epsilon}{xH},$$

$$\Rightarrow \quad \left\|\sum_{r=1}^{m-1} \eta_r^i - \sum_{r=1}^{m-1} \eta_r^j, t\right\| < \frac{\epsilon}{xH}$$

and

$$\inf\left\{K^{\frac{s_n}{p}}: \sup\left\{\mathcal{M}\left(\frac{\| \Delta_m^n \eta_k^i - \Delta_m^n \eta_k^j, t \|}{K}\right)(x_k)^{\frac{1}{s_k}}\right\} \leq 1, K > 0, \forall t \in C_2\right\} < \epsilon \tag{2.1}$$

This implies that $\{\Delta_m^n \eta_1^i\}$ is a Cauchy sequence in C_2. Since C_2 is a modified Banach algebra, then $\{\eta_1^i\}$ converges in C_2 . Let

$$\lim_{n\to\infty} \sum_{r=1}^{m-1} \eta_r^i = \sum_{r=1}^{m-1} \eta_r$$

Thus

$\left\|\sum_{r=1}^{m-1} \eta_r^i - \sum_{r=1}^{m-1} \eta_r \;, t\right\| < \frac{\epsilon}{xH}$ as $i \to \infty$, $\forall\, t \in C_2$.

Now from equation (2.1), we have

$$\mathcal{M}\left(\frac{\| \Delta_m^n \eta_k^i - \Delta_m^n \eta_k^j, t \|}{p(\eta_k^i - \eta_k^j)}\right)(x_k)^{1/s_k} \leq 1$$

This implies that

$$\mathcal{M}\left(\frac{\left\| \Delta_m^n \eta_k^i - \Delta_m^n \eta_k^j, t\right\|}{p(\eta_k^i - \eta_k^j)}\right) \leq (p_k)^{1/t_n} \leq \mathcal{M}\left(\frac{xH}{2}\right)$$

Therefore,

$$\left\| \Delta_m^n \eta_k^i - \Delta_m^n \eta_k^j, t\right\| < \frac{\epsilon}{2}$$

Therefore, $\left\{\Delta_m^n \eta_k^i\right\}$ is a Cauchy sequence in C_2 , for all $k \in \mathbb{N}$. Hence, $\left\{\Delta_m^n \eta_k^i\right\}$ converges in C_2 . Suppose $\lim_{i\to\infty}\left(\Delta_m^n \eta_k^i\right) = \Delta_m^n \xi_k, \forall\, k \in N$. Thus, $\lim_{i\to\infty} \Delta_m^n \eta_2^i = \Delta_m^n \xi_1 - \Delta_m^n \eta_1$ and in general, we have $\lim_{i\to\infty} \Delta_m^n \eta_{k+1}^i = \Delta_m^n \xi_k - \Delta_m^n \eta_k, \forall\, k \in \mathbb{N}$. Hence, by the continuity of the Orlicz function $\mathcal{M}$, we have

$$\lim_{j\to\infty} \sup_k \left\{\mathcal{M}\left(\frac{\| \Delta_m^n \eta_k^i - \Delta_m^n \eta_k^j, t \|}{K}\right)(x_k)^{\frac{1}{s_k}}\right\} \leq 1,$$

This implies that

$$\sup_k \left\{\mathcal{M}\left(\frac{\| \Delta_m^n \eta_k^i - \Delta_m^n \eta_k^j, t \|}{K}\right)(x_k)^{\frac{1}{s_k}}\right\} \leq 1,$$

Let $i \geq k_0$ and by taking infimum for the values of $K > 0$, we obtain $p(\eta^i - \eta) < \epsilon$. So,$\{\eta^i - \eta\} \in \ell^\infty(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$. Hence, $\eta \in \ell^\infty(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$. Therefore, $\ell^\infty(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ is complete.

Corollary 2.1: $\ell^\infty(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ is a Banach space for $s \in \ell^\infty(C_0)$.

Theorem 2.6: $\ell(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$, $c(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ and $c_0(C_2, \|.,.\|, \Delta_m^n, s, \mathcal{M})$ are Banach spaces

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