

# Mathematical Modeling of RLC Circuit Using Theory and Applications of Kalman Filtering

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**Abstract:** *In this research article, an application of continuous Kalman Filtering for an RLC Circuit is presented. In addition of white noise term, the deterministic model of the circuit is changed as stochastic source and the resultant solution is computed using Ito formula, which is the charge of the filtering problem for the RLC circuit.*

**Keywords:** *Stochastic Differential Equation (SDE), Kalman Bucy Filter, White Noise, Ito formula, RLC circuit.*

## 1. INTRODUCTION

Real modeling system of ordinary differential equations (ODEs), ignore the notice of stochastic effects. The differential equations can be change into stochastic differential equation by adding the arbitrary elements and phrase the stochastic equations [1] which gives at least one of the terms is a stochastic process, the resultant is also a stochastic process.

SDEs take part in an appropriate role for numerous application areas such as environmental modeling, engineering, biological modeling, etc. Several researchers used the application of SDEs to investigate the radar scattering and wireless communications. Field and Tough [2,3] have efficiently used SDEs in K-distributed noise in electromagnetic scattering. Charalambous and et al, [4] used SDEs equations to represent multipath fading channels. To prove SDE model they used Meticulous mathematical analysis and computer simulation method. A first-order stochastic auto regressive (AR) model is formed directly with the time variable by discretizing the SDE model. Many researchers have studied in modeling of electrical circuits which is the major application of SDEs. W. Kampowsky and et al, illustrated by applying white noise [5] of electrical circuits to classify and numerical simulate. C. Penski described its application in circuit simulation using new numerical solution for SDEs with white noise [6]. For modeling a series of RC Circuit using different application of noise terms using Ito stochastic calculus including numerical solution was proved by T. Rawat[7].

However, E. Kolarova proved the applications of stochastic integral equations using RL Circuits [8]. On the automated formal confirmation of analog/RF circuits employed by R. Narayanan and et al. The hitch in filtering has an significant branch for the SDEs. Instinctively, an optimal way of observing the problem is to filter the noise terms. In RC and RL circuit the deterministic, stochastic model was found [9, 10]. Basically, the filtering problem based on a series of noisy observations, gives a formula for estimating the state of a system, which resulting the noisy linear differential equations.

## 2. STOCHASTIC DIFFERENTIAL EQUATION

A Stochastic Differential Equation is comprised of differential equation that includes a stochastic process resulting in a solution which is also a stochastic process. Let  $\theta$  be the probability space, and  $R$  is a collection of all  $Y_t, t \in P$  of random variables which represents a stochastic process with state space and also takes values in  $R$  for the Parameter  $P$ . If  $P$  is countable, then the process is discrete otherwise continuous. A stochastic process is to describe the some randomness in an uncorrelated white Gaussian noise.

The general scalar SDE is of the form

$$dX(t) = \alpha(t, X(t))dt + \beta(t, X(t))dV(t) \quad (2.1)$$

where  $\alpha(t, X(t)) : [0, T] \times R \rightarrow R$  and  $\beta(t, X(t)) : [0, T] \times R \rightarrow R$  are drift & diffusion coefficients and  $dV(t)$  is the wiener process.

The integral form of SDE is given by

$$X(t) = x_0 + \int_0^t \alpha(s, X(s))ds + \int_0^t \beta(s, X(s))dV(s) \quad (2.2)$$

Here, the integral  $ds$  is ordinary integral and  $dV(s)$  are stochastic integrals.

Let  $X(t)$  be the solution of SDE (2.1)

Let  $\sigma(t, Y(t)) : (0, \infty) \times R \rightarrow R$  be a double differentiable function.

The function  $U(t) = \sigma(t, Y(t))$  satisfies

$$dU(t) = \frac{\partial \sigma}{\partial t}(t, Y(t)) + \frac{\partial \sigma}{\partial x}(t, Y(t))dX(t) + \frac{1}{2} \frac{\partial^2 \sigma}{\partial x^2}(t, Y(t))(dY(t))^2 \quad (2.3)$$

where,  $d(Y(t))^2 = dY(t).dY(t)$  satisfies the condition

$$dt.dt = dt.dV(t) = dV(t).dt = 0, dV(t).dV(t) = dt$$

## 3. THE THEORY OF KALMAN FILTERING

$$\text{Suppose } \frac{dX(t)}{dt} = a(t, X(t)) + \tau(t, X(t))U(t), t \geq 0 \quad (3.1)$$

where  $a$  and  $\rho$  satisfy conditions

$$|a(t, x)| + |\tau(t, x)| \leq K(1 + |x|); |a(t, x) - a(t, y)| + |\tau(t, x) - \tau(t, y)| \leq M|x - y|; x, y \in R, t \in [0, T]$$

For some constant  $K, M$  and  $U(t)$  is white noise.

**Theorem 3.1:** The one-dimensional Kalman Bucy filter

The 1-dimensional solution  $\hat{X}(t)$  of the linear filtering problem

Linear system:

$$dX(t) = E(t)X(t)dt + F(t)X(t)dU(t); E(t), F(t) \in R \quad (3.2)$$

Linear Observation:

$$dZ(t) = L(t)X(t) dt + N(t)dV(t); L(t), N(t) \in R \quad (3.3)$$

which satisfies the SDE

$$d\hat{X}(t) = \left( E(t) - \frac{L^2(t)S(t)}{N^2(t)} \right) \hat{X}(t)dt + \frac{L(t)S(t)}{N^2(t)} dZ(t); \hat{X}(0) = E[X(0)] \quad (3.4)$$

where,  $S(t) = E \left[ \left( X(t) - \hat{X}(t) \right)^2 \right]$  gratifies the Riccati equation

$$\frac{dS}{dt} = 2E(t)S(t) - \frac{L^2(t)S^2(t)}{E^2(t)} + F^2(t), S(0) = E \left[ \left( X(0) - \hat{X}(0) \right)^2 \right] \quad (3.5)$$

#### 4. THE DETERMINISTIC MODEL OF RLC CIRCUIT

Every RLC circuit has some finite resistance R, inductance L and capacitance C. Here, Q(t) represents the charge of the circuit which is a fixed point at time t and U(t) is the potential source at time t. According to Kirchhoff's law satisfies the differential equation

$$-U(t) + RI(t) + L\frac{dI}{dt} + \frac{1}{C} \int I(t)dt = 0 \quad (4.1)$$

Where  $I(t) = \frac{dQ(t)}{dt}$ , then

$$LQ''(t) + RQ'(t) + \frac{1}{C} Q(t) = U(t) \quad (4.2)$$

With initial conditions  $Q(0) = Q_0, Q'(0) = I_0$ .

We introduce the vector form in equation (4.2)

$$\text{Here } X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} Q(t) \\ Q'(t) \end{pmatrix}$$

To the system,

$$X_1' = X_2$$

$$X_2' = \frac{-1}{CL} X_1 - \frac{R}{L} X_2 + \frac{U(t)}{L} \quad (4.3)$$

In matrix form,

$$\frac{dX(t)}{dt} = AX(t) + K(t), X(0) = X_0 \quad (4.4)$$

$$\text{where } A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}, K(t) = \begin{pmatrix} 0 \\ \frac{U(t)}{L} \end{pmatrix}, X_0 = \begin{pmatrix} Q_0 \\ I_0 \end{pmatrix}$$

Using Ito formula

$$f(t, X(t)) = e^{-At} X(t) \quad (4.5)$$

$$df(t, X(t)) = d(e^{-At} X(t))$$

$$= e^{-At} X(t)(-A) dt + e^{-At} dX(t)$$

$$= -Ae^{-At} X(t)dt + e^{-At} [AX(t)dt + K(t)dt]$$

$$= e^{-At} K(t)dt$$

Integrating, we get,

$$\int_0^t dg(e^{-As} X(s)) = \int_0^t e^{-As} K(s) ds$$

$$e^{-At} X(t) - X(0) = \int_0^t e^{-As} K(s) ds$$

$$X(t) = e^{At} X_0 + \int_0^t e^{A(t-s)} K(s) ds \quad (4.6)$$

which is the analytic solution of equation (4.4)

#### 5. COMPUTATION RLC CIRCUIT WITH STOCHASTIC SOURCE

Now, include some randomness in the potential source

$$U^*(t) = U(t) + \text{noise}$$

$$U^*(t) = U(t) + \alpha \xi(t) \quad (5.1)$$

where  $\xi(t)$  is a white noise process of mean zero and variance one and  $\alpha$  is intensity of noise with non-negative constant.

$$\frac{dX(t)}{dt} = AX(t) + K(t) + \begin{bmatrix} 0 \\ \frac{\alpha}{L} \xi(t) \end{bmatrix}$$

$$dX(t) = (AX(t) + K(t)) dt + \begin{bmatrix} 0 \\ \frac{\alpha}{L} \xi(t) dt \end{bmatrix} \quad (5.2)$$

Now, Instead  $\xi(t)dt$  by  $dW(t)$ . Usually, for the wiener process  $W(t)$ , the time derivative is the white noise.

$$dX(t) = (AX(t) + K(t)) dt + \begin{bmatrix} 0 \\ \frac{\alpha}{L} dW(t) \end{bmatrix}$$

$$dX(t) = [AX(t) + K(t)]dt + cdW(t) \tag{5.3}$$

where  $c = \begin{bmatrix} 0 \\ \frac{\alpha}{L} \end{bmatrix}$

By using Ito formula in equation (5.3), we get,

$$\begin{aligned} dh(t, X(t)) &= d(e^{-At}X(t)) \\ &= e^{-At}X(t)(-A) dt + e^{-At}dX(t) \\ &= -Ae^{-At}X(t)dt + e^{-At}[AX(t)dt + K(t)dt + cdW(t)] \\ &= e^{-At}[K(t) dt + cdW(t)] \end{aligned}$$

Integrating, we get,

$$\begin{aligned} \int_0^t dh(e^{-As}X(s))ds &= \int_0^t e^{-As}K(s)ds + c \int_0^t e^{-As}dW(s) \\ e^{-At}X(t) - X(0) &= \int_0^t e^{-At}K(t)dt + c \int_0^t e^{-As}dW(s) \\ X(t) &= e^{At}X(0) + \int_0^t e^{A(t-s)}K(s)ds + c \int_0^t e^{A(t-s)}dW(s) \end{aligned} \tag{5.4}$$

Equation (5.4) is the equation of the stochastic source and also it is a random process. For every  $t > 0$  and

If  $E[X(0)X^T(0)] < \infty$ , the expectation  $E[X(t)] = m(t)$  is the solution of the ordinary differential equation

$$\frac{dm(t)}{dt} = Am(t) + E[K(t)]$$

Taking the expectation of equation (5.4) we have,

$$E[X(t)] = e^{At}E[X(0)] + E\left(\int_0^t e^{A(t-s)}K(s)ds\right) + c E\left(\int_0^t e^{A(t-s)}dW(s)\right)$$

Since,  $E\left(\int_0^t g(s)dVs\right) = 0$ , we get

$$m(t) = E[X(t)] = e^{At}E[X(0)] + E\left(\int_0^t e^{A(t-s)}K(s)ds\right) \tag{5.5}$$

For every  $t > 0$  and if the random variable  $X(0) = \text{constant}$ , we get  $m(t) = E[X(t)]$  which is independent of the oscillation part of the SDE.

$$\begin{aligned} E[X^2(t)] &= E\left[e^{At}X(0) + \int_0^t e^{A(t-s)}a(s)ds + c \int_0^t e^{A(t-s)}dW(s)\right]^2 \\ &= e^{2At}E[X^2(0)] + E\left[\int_0^t e^{A(t-s)}a(s)(ds)\right]^2 + c^2E\left[\int_0^t e^{2A(t-s)}(dW(s))^2\right] \end{aligned}$$

Since  $E[(dW(t))^2] = dt$  &  $E[ds \cdot ds] = 0$ , we get

$$\begin{aligned} E[X^2(t)] &= e^{2At}E[X^2(0)] + c^2e^{2At}E\left[\int_0^t e^{-2As}ds\right] \\ &= e^{2At}E[X^2(0)] + c^2e^{2At}\left[\frac{e^{-2As}}{-2A}\right]_0^t \\ &= e^{2At}E[X^2(0)] - \frac{c^2}{2A}e^{2At}[e^{-2At} - 1] \\ E[X^2(t)] &= e^{2At}E[X^2(0)] - \frac{c^2}{2A}[1 - e^{2At}] \end{aligned} \tag{5.6}$$

$$Var(X(t)) = E[X^2(t)] - E[X(t)]^2$$

$$\begin{aligned} Var(X(t)) &= e^{2At}E[X^2(0)] - \frac{c^2}{2A}[1 - e^{2At}] - e^{2At}E[X(0)]^2 + E\left[\int_0^t e^{2A(t-s)}a(s)(ds)\right]^2 \\ VarX(t) &= e^{2At}Var(X(0)) - \frac{c^2}{2A}[1 - e^{2At}] \end{aligned} \tag{5.7}$$

Corollary 5.1

Let  $A = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix}$  then  $e^{At} = e^{-\lambda t} \xi \{ (\xi \cos(\xi t) + \lambda \sin(\xi t) I + A \sin(\xi t)) \}$

Where  $\lambda = \frac{b}{2}, \xi = \sqrt{a - \frac{b^2}{4}}$

By corollary 5.1, we have  $E[X(t)] \rightarrow 0$  and  $Var(X(t)) \rightarrow \frac{c^2}{2A}$  as  $t \rightarrow \infty$ . Therefore, the distribution of  $X(t)$  approaches to  $N(0, \frac{c^2}{2A})$  as  $t \rightarrow \infty$ .

Then, based on the properties of normal distribution,  $X(t)$  is a Gaussian process.

For any value of  $t$ ,

$$P(|X(t) - m(t)| < 1.96 * \alpha = 0.95)$$

Using this condition, we are able to compute  $m(t)$  and  $\alpha(t)$  and we can easily predict the probability with 95% confident interval  $(m(t) - \delta, m(t) + \delta)$ , which gives the trajectories of the stochastic solution.

From Kalman Bucy Filter

$$\text{Linear Observation: } dZ(t) = Q(t)dt + dU(t) \quad (5.8)$$

$$\text{Linear System: } dX_2(t) = \frac{-1}{CL} X_1(t) - \frac{R}{L} X_2(t) + \frac{U(t)}{L}$$

$$d\hat{X}(t) = \left( \frac{-1}{CL} - S(t) \right) \hat{X}(t)dt + S(t) dZ(t); \hat{X}(0) = E[X(0)] = 0 \quad (5.9)$$

Here,  $S(t) = E \left[ \left( X(t) - \hat{X}(t) \right)^2 \right]$  satisfies the Riccati equation,

$$S'(t) = \frac{-2}{LC} S(t) - S^2(t) + \frac{1}{L^2}, \quad S(0) = E[(X_0)^2] = a^2 \quad (5.10)$$

$$S(t) = M(t) - \frac{1}{LC}, \quad S'(t) = M'(t)$$

Squaring we get,

$$S^2(t) = \left( M(t) - \frac{1}{LC} \right)^2 = M^2(t) + \frac{1}{L^2 C^2} - \frac{2M(t)}{LC}$$

Which is the ordinary differential equation for the function  $M(t)$

$$\begin{aligned} S'(t) &= \frac{-2}{LC} S(t) - S^2(t) + \frac{1}{L^2} \\ &= \frac{-2}{LC} \left( M(t) - \frac{1}{LC} \right) - \left( M^2(t) + \frac{1}{L^2 C^2} - \frac{2M(t)}{LC} \right) + \frac{1}{L^2} \\ &= \frac{1}{L^2 C^2} M^2(t) + \frac{1}{L^2} \end{aligned}$$

$$S'(t) = \left( \frac{1+C^2}{L^2 C^2} \right) M^2(t)$$

$$\text{But } S'(t) = M'(t) = \left( \frac{1+C^2}{L^2 C^2} \right) M^2(t)$$

$$\frac{dM(t)}{dt} = v^2 - M^2(t), \quad \text{where } v^2 = \frac{1+C^2}{L^2 C^2}$$

$$\frac{dM(t)}{dt} = \frac{1}{v^2 - M^2(t)}$$

The implicit solution is

$$\frac{-1}{2} \ln \left| \frac{M(t) - v}{M(t) + v} \right| = b + r$$

$$M(t) = v \frac{e^{-2(b+r)+1}}{e^{-2(b+r)-1}}$$

$$\text{Then, } S(t) = M(t) - \frac{1}{LC} = v \frac{e^{-2(b+r)+1}}{e^{-2(b+r)-1}} - \frac{1}{LC}$$

However,  $r = \frac{1}{2} Ln \left| \frac{S(0) + \frac{1}{LC} + v}{S(0) + \frac{1}{LC} - v} \right|$  and for large value  $b$ , we have  $M(t) \approx v$  and  $S(t) = v - \frac{1}{LC}$

We substitute this equation in equation (4.14) we get the SDE for the filter

$$d\hat{X}(t) = \left( \frac{-1}{CL} - v + \frac{1}{LC} \right) \hat{X}(t)dt + \left( v - \frac{1}{LC} \right) dZ(t); \hat{X}(0) = E[X(0)] = 0$$

$$d\hat{X}(t) = -v\hat{X}(t)dt + \left(v - \frac{1}{LC}\right)dZ(t) \quad (5.11)$$

Using Ito formula for the function

$$G(t, \hat{X}(t)) = e^{vt}\hat{X}(t)$$

$$\begin{aligned} d(e^{vt}\hat{X}(t)) &= ve^{vt}\hat{X}(t)dt + e^{vt}d\hat{X}(t) \\ &= ve^{vt}\hat{X}(t)dt + e^{vt}[-v\hat{X}(t)dt + \left(v - \frac{1}{LC}\right)dZ(t)] \\ &= e^{vt}\left(v - \frac{1}{LC}\right)dZ(t) \end{aligned}$$

Integrating, we get

$$\begin{aligned} e^{vt}\hat{X}(t) &= \int_0^t e^{vs}\left(v - \frac{1}{LC}\right)dZ(s) \\ \hat{X}(t) &= \left(v - \frac{1}{LC}\right)\int_0^t e^{v(s-t)}dZ(s) \end{aligned}$$

This is the Filtering problem of the charge  $\hat{X}(t)$  of the RLC circuit under stochastic source.

## 6. CONCLUSION

In this paper the theory and application of Kalman Filtering is used to the deterministic model and computed the stochastic source of RLC Circuit. The filtering problem of RLC Circuit under stochastic source is found.

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