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On Sequence Spaces Defined Via Euler And Matrix Transformations

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ABSTRACT: We present the topological properties of some lacunary sequence spaces on n-normed space defined via Euler and matrix summability transformations. Further some inclusion relation between these spaces are studied.

INTRODUCTION AND PRELIMINARIES

Euler summability is widely used in numerical analysis to improve the convergence of the series. These techniques are useful in computer science especially in making graphics and the accelerated convergence techniques are also used to find eigenvalues and eigenvectors of dynamical systems. The Euler transform (E,p) of the sequence $S=(s_n)$ of the partial sums of a series $\sum_{k=0}^{\infty} a_k$ is defined as $E_n^p(S)=\frac{1}{(1+p)^n}\sum_{v=0}^{n}\binom{n}{v}p^{n-v}s_v, p$ is positive real. A series $\sum a_n$ is called (E,p) −summable to s if $E_n^p(S)=\frac{1}{(1+p)^n}\sum_{v=0}^{n}\binom{n}{v}p^{n-v}s_v \to s$ and it is called absolutely $(E,p)-$ summable if $\sum_k\left|E_k^p(S)-E_{k-1}^p(S)\right|<\infty$. Suppose $x=(x)_n$ be the sequence of scalars, for $k\geq 1$, we will represent $N_n(x)$ the difference $E_n^p(x)-E_{n-1}^p(x)$, where E_n^p is defined as above. Through the use of Abel's transform we can write

$$N_n(x)=-\frac{1}{(1+p)^{n-1}}\sum_{j=0}^{n-2}x_{j+1}A_j+\frac{s_{n-1}A_{n-1}}{(1+p)^{n-1}}+\frac{s_n}{(1+p)^n}-\frac{p^{n-1}}{(1+p)^n}s_0,$$

Where $A_k=\sum_{j=0}^{k}\left[\frac{p}{p+1}\binom{n}{j}-\binom{n-1}{j}\right]p^{n-j-1}$. For a sequence $x=(x_n), y=(y_n)$ and the scalar λ, we have $N_n(x+y)=N_n(x)+N_n(y)$ and $N_n(\lambda x)=\lambda N_n(x)$.
In this paper we will study the topological properties of the class of sequence space defined using
Musielak-Orlicz function. We introduce these spaces by using the Euler and matrix transformations. Finally, we present the application of these spaces to statistical convergence. Before proceeding we discuss some basic definitions and results required for the further discussion.
A non-decreasing , continuous and convex function M with $M(0)=0, M(x)>0\ for\ x>0\ and\ M(x)\to\infty\ as\ x\to\infty$ is called an Orlicz function. Orlicz sequence space [10] denoted l_M is
the space of sequences $x=(x_n)$ which satisfy $\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)<\infty, \rho>0$.

Theorem 1.1. $(l_M,\|x\|)$ is a Banach space, where $\|x\|=inf\left\{\rho>0:\sum_{j=1}^{\infty}M\left(\frac{|x_j|}{\rho}\right)\leq 1\right\}$.

A sequence of Orlicz functions is called Musielak-Orlicz function. For more information about the complementary function of a Musielak-Orlicz function, Musielak-Orlicz sequence space,Luxemburg norm we refer the reader to [10].

A real-valued function $\|.,\dots,.\|$ Defined on X^n,where X is a linear space over the (real or complex) field $\mathbb{K}$ of dimension $d, d \geq n \geq 2, n \in \mathbb{N}$ is called an n-norm ([5],[9]) if it fulfils the conditions given below

(1) $\|x_1, x_2, x_3 \dots x_{n-1}, x_n\| = 0$ iff $x_1, x_2, \dots, x_n$ are linearly independent;
(2) $\|x_1, x_2, x_3 \dots, x_{n-1}, x_n\|$ is invariant under permutations;
(3) $\|\alpha x_1, x_2, x_3 \dots, x_{n-1}, x_n\| = |\alpha|\|x_1, x_2, \dots, x_n\| \; \forall\, \alpha \in \mathbb{K}$;
(4) $\|x + x', x_2, x_3 \dots, x_{n-1}, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$.

The pair $(X(\mathbb{K}), \|.,\dots,.\|)$ Is called an n-normed space. For further detail about n-normed space,see [6] and [7].

A sequence $\theta = (k_r)$ of positive integers with $k_0 = 0, 0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ is called a Lacunary sequence [4]. The intervals $I_r = (k_{r-1}, k_r]$ are determined by θ and the ratio $\frac{k_r}{k_{r-1}}$ are denoted as q_r.

For more detail about sequence spaces we refer the reader to [1],[2] and [11].

Let $\mathfrak{M} = [m_{ij}] = \begin{bmatrix} p_1 & w_1^{(1)} & w_1^{(2)} & \dots \\ w_1^{(-1)} & p_2 & w_2^{(1)} & \dots \\ w_1^{(-2)} & w_2^{(-1)} & p_3 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$ where $p = (p_i)$ and $w^{(t)} = (w_i)^{(t)}$ are some fixed numerical sequences, $t \in \mathbb{Z}\backslash\{0\}$ .For a fixed $k_f \in \mathbb{N}$ we define a finite sequence t_n with k_f terms as $t_n = \begin{cases} \frac{n+1}{2}, n \; is \; odd \\ \frac{-n}{2}, n \; is \; even \end{cases}$. We construct a matrix $\mathfrak{M}_{(p,w^t,k_f)} = \mathfrak{M}, w^{t_i} = 0 \;\forall\, i > k_f$ and for $i = 1,2,\dots,k_f$ we have some fixed sequences w^{t_i} and p.

Example 2.1. For $k_f = 2$ we have $t_1 = 1, t_2 = -1$, we define $p_i = -1 \;\forall i$ and $w_i^{(t)} = \begin{cases} 1, for \; t = -1,1 \\ 0, \forall t \in \mathbb{Z}\backslash\{0,1,-1\} \end{cases}$ then we have $\mathfrak{M}_{(p,w^t,2)} x = \langle \sum_{j=1}^{\infty} m_{ij} \xi_j \rangle_n = \langle -\xi_1 + \xi_2, \xi_1 - \xi_2 + \xi_3, \xi_2 - \xi_3 + \xi_4, \xi_3 - \xi_4 + \xi_5 \dots \rangle$.

For a Musielak-Orlicz function $\mathcal{M} = (M_j)$, we define the sequence given below in this paper:

$$E_n^q\left(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,j_f)}, \|.,\dots,.\|\right) = \left\{ x = (x_j) : \lim_r \frac{1}{h_r} \sum_{j=1}^{\infty} k^{-s} \left[M_j \left(\left\| \frac{u_j N_{j\left(\mathfrak{M}_{(p,w^t,j_f)} x\right)}}{\rho}, z_1, \dots z_{n-1} \right\| \right) \right]^{p_j} < \infty, s \geq 0, for \; some \; \rho > 0 \right\}$$

Here,$p = (p_j)$ is a bounded sequence of non-negative reals and $u = (u_j)$ a sequence of positive reals.

Lemma 2.1(Maddox,[8]). If $0 \leq p_j \leq \sup p_j = H, K = max(1, 2^{H-1})$ then $|a_j + b_j|^{p_j} \leq K\{|a_j|^{p_j} + |b_j|^{p_j}\}$ for all $j \; and \; a_j, b_j \in \mathbb{C}$.Also $|a|^{p_j} \leq max(1, |a|^H) \; for \; all \; a \in \mathbb{C}$.

Theorem 2.1: $E_n^q\left(\mathcal{M},u,p,s,\mathfrak{M}_{(p,w^t,j_f)},\|.,\dots,.\|\right)$ Is a linear space over the field of complex numbers and is paranormed by the paranorm $g(x)$ defined as:

$$g(x)=inf\left\{\rho^{p_n/H}:\left(\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(\left\|\frac{u_kN_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho},z_1,\dots z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\leq 1, n=1,2,3,\dots\right\},$$

Here $H=\max(1,sup_k p_k)$

Proof: Let $x=(x_k),y=(y_k)\in E_n^q\left(M,u,p,s,\mathfrak{M}_{(p,w^t,k_f)},\|.,\dots,.\|\right),\alpha,\beta\in\mathbb{C}$. There will exist positive integer ρ_1,ρ_2 so that,

$$\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(\left\|\frac{u_kN_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho_1},z_1z_2,\dots,z_{n-1}\right\|\right)\right]^{p_k}<\infty \quad \text{and}$$

$$\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(\left\|\frac{u_kN_k\left(\mathfrak{M}_{(p,w^t,k_f)}y)\right)}{\rho_2},z_1z_2,\dots,z_{n-1}\right\|\right)\right]^{p_k}<\infty.$$

Since (M_k) is non-decreasing convex function so that $\max(2|\alpha|p_1,2|\beta|\rho_2=\rho_3$

We have
$$\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(\left\|\frac{u_kN_k\left(\mathfrak{M}_{(p,w^t,k_f)}(\alpha x+\beta y)\right)}{\rho_3},z_1z_2,\dots,z_{n-1}\right\|\right)\right]^{p_k}\leq$$
$$\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(\left\|\frac{u_kN_k\left(\mathfrak{M}_{(p,w^t,k_f)}\alpha x\right)}{\rho_3},z_1z_2,\dots,z_{n-1}\right\|\right)\right]^{p_k}+$$
$$\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(\left\|\frac{u_kN_k\left(\mathfrak{M}_{(p,w^t,k_f)}\beta y\right)}{\rho_3},z_1z_2,\dots,z_{n-1}\right\|\right)\right]^{p_k}\leq$$
$$K\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(\left\|\frac{u_kN_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho_3},z_1z_2,\dots,z_{n-1}\right\|\right)\right]^{p_k}+$$
$$K\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty}k^{-s}\left[M_k\left(\left\|\frac{u_kN_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho_3},z_1z_2,\dots,z_{n-1}\right\|\right)\right]^{p_k}<\infty.$$

Therefore $\alpha x+by\in E_n^q\left(M,u,p,s,\mathfrak{M}_{(p,w^t,k_f)},\|.,\dots,.\|\right)$. Hence the space is linear. Further we have $g(x)=g(-x)$ and $g(x+y)\leq g(x)+g(y)$ and $M_k(0)=0$,we have $inf\{\rho^{p_n/H}\}=0$ if $x=0$. Let us take a number λ, then,

$g(\lambda x) = inf\left\{\rho^{p_n/H}: \lim_r \frac{1}{h_r}\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{\lambda u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1 z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \leq 1, n = 1,2,3, \dots\right\}$ implies that

$$g(\lambda x) = inf\left\{(\lambda s)^{p_n/H}: \lim_r \frac{1}{h_r}\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{\lambda u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1 z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \leq 1, n = 1,2,3, \dots\right\}$$

$s = \frac{\rho}{|\lambda|}$.By 2.1 we have $|\lambda|^{p_k/H} \leq (\max(1, |\lambda|^H)^{\frac{1}{H}}$ and hence,

$$g(\lambda x) = (\max(1, |\lambda|^H)^{\frac{1}{H}} inf\left\{(s)^{p_n/H}: \lim_r \frac{1}{h_r}\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{\lambda u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1 z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} \leq 1, n = 1,2,3, \dots\right\} H \leq 1, N = 1,2, \dots$$

Clearly Clearly $g(x) \to 0$ when $x \to 0$ in $E_n^q\left(M, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right)$.Now let, $\lambda_n \to 0\ as\ \ n \to \infty$ and $E_n^q\left(M, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right)$ For any $\epsilon > 0$, let n_0 be a positive integer such that

$$\lim_r \frac{1}{h_r}\sum_{k=n_0+1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1 z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} < \frac{\epsilon}{2}$$

For some $\rho > 0$.which implies that

$$\left(\lim_r \frac{1}{h_r}\sum_{k=n_0+1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1 z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq \frac{\epsilon}{2}.$$

Let us take $0 < |\lambda| < 1$, then convexity of (M_k) implies

$$\lim_r \frac{1}{h_r}\sum_{k=n_0+1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{\lambda u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1 z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} <$$

$$|\lambda| \lim_r \frac{1}{h_r}\sum_{k=n_0+1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1 z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} < \left(\frac{\epsilon}{2}\right)^H.$$

Since (M_k) is continuous everywhere in $[0, \infty)$,so

$$h(t) = \lim_r \frac{1}{h_r}\sum_{k=1}^{n_0} k^{-s}\left[M_k\left(\left\|\frac{t u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1 z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k}$$

is also continuous at 0. Hence there exists $0 < \delta < 1$ such that $|h(t)| < \epsilon/2$ for some $0 < t < \delta$.Let K is such that $|\lambda_n| < \delta$ for $n > K$ we have

$$\left(\lim_r \frac{1}{h_r}\sum_{k=1}^{n_0} k^{-s}\left[M_k\left(\left\|\frac{\lambda_n u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1 z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} < \frac{\epsilon}{2}.$$

Thus for $n > K$

$$\left(\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{\lambda_n u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1 z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} < \epsilon$$

Hence,$g(\lambda x)$ converges to 0 as λ converges to 0 and hence the result. ■

Theorem-2.2: For the Musielak-Orlicz functions $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ and for s, s_1, s_2, where s, s_1, s_2 are nonnegative real numbers, we have

(i) $E_n^q\left(\mathcal{M}', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right) \cap E_n^q\left(\mathcal{M}'', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right) \subseteq E_n^q\left(\mathcal{M}' + \mathcal{M}'', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right)$,

(ii) If $s_1 \le s_2$,then
$E_n^q\left(\mathcal{M}', u, p, s_1, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right) \subseteq E_n^q\left(\mathcal{M}', u, p, s_2, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right)$,

(iii) If $\mathcal{M}'$ is equivalent to $\mathcal{M}''$,then
$E_n^q\left(\mathcal{M}', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right) = E_n^q\left(\mathcal{M}', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right)$.

Proof: Proof of this theorem is routine. ■

Theorem:2.3: $E_n^q\left(\mathcal{M}, u, r, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right) \subseteq E_n^q\left(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right), 0 < r_k \le p_k < \infty$.

Proof: Let $x \in E_n^q\left(\mathcal{M}, u, r, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right)$.There exists some $\rho > 0$ such that

$$\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{r_k} < \infty.$$

Hence, $M_k\left(\left\|\frac{u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right) \le 1$ for sufficiently large k, let $k \ge k_0$ for some fixed $k_0 \in \mathbb{N}$. As (M_k) is non-decreasing so

$$\lim_r \frac{1}{h_r}\sum_{k\ge k_0}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k}$$

$$\le \lim_r \frac{1}{h_r}\sum_{k\ge k_0}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_k N_k\left(\mathfrak{M}_{(p,w^t,k_f)}x\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{r_k} < \infty.$$

Hence $x \in E_n^q\left(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right)$. ■

Theorem-2.4:If $0 < p_k \le 1$,then

$$E_n^q\left(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right) \subseteq E_n^q\left(\mathcal{M}, u, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right).$$

(ii) If $p_k \ge 1$ for all k,then

$$E_n^q\left(\mathcal{M}, u, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right) \subseteq E_n^q\left(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \|., \dots, .\|\right).$$

Proof- Proof is similar to that of 2.3. ■

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