Maximum Modulus of a Polynomial and its Derivative

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Abstract- Let $p(z)$ be a polynomial of degree n having no zero zero in $|z| \le k$, $k \ge 1$, then for $1 \le R \le k$, Dewan and Bidkham [J. Math. Anal. Appl., 166(1992), 319-324] proved

$$\max_{|z|=R}\left|p'(z)\right| \le n\frac{(R+k)^{n-1}}{(1+k)^{n}}\max_{|z|=1}\left|p(z)\right|.$$

The result is best possible and extremal polynomial is $p(z)=(z+k)^n$.

In this paper, by involving certain coefficients of the polynomial $p(z)$, we prove a result concerning the estimate of maximum modulus of derivative of $p(z)$, which not only improves as well as generalizes the above result, but also has interesting consequences.

Keywords – Polynomial, Polynomial Inequalities, Maximum Modulus.

I. Introduction

It was for the first time, Bernstein [9, 12] investigated an upper bound for the maximum modulus of the first derivative of a complex polynomial on the unit circle in terms the maximum modulus of the polynomial on the same circle and proved the following famous result known as Bernstein's inequality that if $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}\left|p'(z)\right| \le n\max_{|z|=1}\left|p(z)\right|. \tag{1.1}$$

Inequality (1.1) is best possible and equality occurs for $p(z)=\lambda z^n$, $\lambda \ne 0$, is any complex number.

If we restrict to the class of polynomials having no zero in $|z|<1$, then inequality (1.1) can be sharpened as

$$\max_{|z|=1}\left|p'(z)\right| \le \frac{n}{2}\max_{|z|=1}\left|p(z)\right|. \tag{1.2}$$

The result is sharp and equality holds in (1.2) for $p(z)=\alpha+\beta z^n$, where $|\alpha|=|\beta|$.

Inequality (1.2) was conjectured by Erdös and later proved by Lax [7].

It was asked by Professor R.P. Boas that if $p(z)$ is a polynomial of degree n not vanishing in $|z|<k$, $k>0$, then how large can

$$\left\{ \max_{|z|=1}\left|p'(z)\right| \Big/ \max_{|z|=1}\left|p(z)\right| \right\} \quad \text{be?} \tag{1.3}$$

A partial answer to this problem was given by Malik [8], who proved

Theorem A. *If $p(z)$ is a polynomial of degree n having no zero in the disc $|z|<k$, $k\geq 1$, then*

$$\max_{|z|=1}\left|p'(z)\right| \leq \frac{n}{1+k}\max_{|z|=1}\left|p(z)\right| \tag{1.4}$$

The result is best possible and equality holds for $p(z)=(z+k)^n$.

For the class of polynomials not vanishing in $|z|<k, k\leq 1$, the precise estimate for maximum of $\left|p'(z)\right|$ on $|z|=1$, in general, does not seem to be easily obtainable.

Dewan and Bidkham [2] generalized Theorem A by considering any circle that lies in a closed circular annulus of radii 1 and k where $k\geq 1$.

Theorem B. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then*

$$\max_{|z|=R}\left|p'(z)\right| \leq n\frac{(R+k)^{n-1}}{(1+k)^n}\max_{|z|=1}\left|p(z)\right|, \text{ for } 1\leq R\leq k. \tag{1.6}$$

The result is best possible and extremal polynomial is $p(z)=(z+k)^n$.

Dewan and Mir [4] further improved and generalized Theorem B.

Theorem C. *If $p(z)=\sum_{\upsilon=0}^{n}a_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then $0<r\leq\rho\leq k$*

$$\max_{|z|=\rho}\left|p'(z)\right| \leq n\frac{(\rho+k)^{n-1}}{(k+r)^n}\left[1-\frac{k(k-\rho)\left(n|a_0|-k|a_1|\right)n}{\left(k^2+\rho^2\right)n|a_0|+2k^2\rho|a_1|}\left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right]\max_{|z|=r}\left|p(z)\right|. \tag{1.7}$$

In this paper, by considering a more general class of polynomials $p(z)$ and involving $\min_{|z|=k}\left|p(z)\right|$, we prove an interesting result, which improves as well as generalizes inequality (1.7) by considering maximum modulus on two different circles lying both inside and on any circle. More precisely, we have

Theorem. *If $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k, k>0$, then for $0<r\leq\rho\leq k$,*

$$\max_{|z|=\rho}\left|p'(z)\right| \le \frac{n\rho^{n-1}}{\rho^{\mu}+k^{\mu}}\left(\frac{\rho^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}\left[1-S\left\{1-\left(\frac{r^{\mu}+k^{\mu}}{\rho^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}\right\}\right]M(p,r) - \frac{n\rho^{n-1}}{\rho^{\mu}+k^{\mu}}\left[I_1+\left\{\frac{nr^{n-1}}{k^n}A_t+1\right\}\min_{|z|=k}\left|p(z)\right|\right], \tag{1.8}$$

where

$$S=\frac{(k-\rho)\left(n|a_0|-\mu|a_{\mu}|k\right)k^{\mu}}{\left(\rho^{\mu+1}+k^{\mu+1}\right)n|a_0|+\mu|a_{\mu}|\left(k^{\mu+1}\rho^{\mu}+k^{2\mu}\rho\right)} \text{ and}$$

$$I_1=n\min_{|z|=k}\left|p(z)\right|\int_r^{\rho}A_t\left\{\left(\frac{t^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}-1\right\}dt \text{ with } A_t=\frac{n|a_0|t^{\mu}+\mu|a_{\mu}|k^{\mu+1}t^{\mu-1}}{\left(t^{\mu+1}+k^{\mu+1}\right)n|a_0|+\mu|a_{\mu}|\left(k^{\mu+1}t^{\mu}+k^{2\mu}t\right)}.$$

The result is best possible for $p(z)=\left(z^{\mu}+k^{\mu}\right)^{\frac{n}{\mu}}$ *where n is a multiple of* μ.

Remark 1.1. Putting $\mu=1$, our Theorem reduces to

Corollary 1.1. *If* $p(z)=\sum_{\upsilon=0}^{n}a_{\upsilon}z^{\upsilon}$ *is a polynomial of degree n having no zero in* $|z|<k, k>0$, *then for* $0<r\le\rho\le k$,

$$\max_{|z|=\rho}\left|p'(z)\right|\le\frac{n(\rho+k)^{n-1}}{(r+k)^n}\left[1-\frac{(k-\rho)\left(n|a_0|-|a_1|k\right)}{\left(\rho^2+k^2\right)n|a_0|+2k^2\rho|a_1|}\left\{1-\left(\frac{r+k}{\rho+k}\right)^n\right\}\right]M(p,r)-\frac{n}{\rho+k} \times\left[I+\left\{\frac{nr^{n-1}}{k^n}\frac{\left(n|a_0|k^2+|a_1|k^2\rho\right)(\rho-r)}{\left(\rho^2+k^2\right)n|a_0|+2k^2\rho|a_1|}+1\right\}\min_{|z|=k}\left|p(z)\right|\right], \tag{1.9}$$

where

$$I=\min_{|z|=k}\left|p(z)\right|\left[n\int_r^{\rho}\left\{\frac{n|a_0|t+|a_1|k^2}{\left(t^2+k^2\right)n|a_0|+2|a_1|k^2t}\right\}\left(\frac{t+k}{r+k}\right)^n dt-\frac{n}{2}In\left\{\frac{\left(r^2+k^2\right)n|a_0|+2|a_1|k^2r}{\left(\rho^2+k^2\right)n|a_0|+2|a_1|k^2\rho}\right\}\right].$$

The result is best possible and equality holds in (1.9) *for* $p(z)=(z+k)^n$.

Remark 1.2. If we make use of the fact of Remark 1.6 to Corollary 1.1, we have the following result, which is an improvement of Theorem C along with the extension of value of k from $k\ge1$ to $k>0$.

Corollary 1.2. *If* $p(z)=\sum_{\upsilon=0}^{n}a_{\upsilon}z^{\upsilon}$ *is a polynomial of degree n having no zero in* $|z|<k, k>0$, *then for* $0<r\le\rho\le k$,

$$\max_{|z|=\rho}\left|p'(z)\right| \le \frac{n(\rho+k)^{n-1}}{(r+k)^n}\left[1-\frac{(k-\rho)\left(n|a_0|-|a_1|k\right)n}{\left(\rho^2+k^2\right)n|a_0|+2k^2\rho|a_1|}\left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right]M(p,r)-\frac{n}{\rho+k}$$
$$\times\left[I+\left\{\frac{nr^{n-1}}{k^n}\frac{\left(n|a_0|k^2+|a_1|k^2\rho\right)(\rho-r)}{\left(\rho^2+k^2\right)n|a_0|+2k^2\rho|a_1|}+1\right\}\min_{|z|=k}\left|p(z)\right|\right], \tag{1.10}$$

where I *is as defined in Corollary* 1.1.

The result is best possible and equality holds in (1.10) *for* $p(z)=(z+k)^n$.

Remark 1.3. Both the above corollaries are more improved versions of Theorem B. Moreover, when we assign $r=\rho=1$, they give improved bounds than given by the well-known inequality (1.4) due to Malik [8].

Remark 1.4. Further, when $k=r=\rho=1$ the above corollaries are improvements of well-known inequality (1.2) proved by by Lax [6].

II LEMMA

For the proofs of the theorems the following lemmas are required.

The first lemma is due to Dewan et. al [3].

Lemma 2.1. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$, $1\le\mu<n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\ge1$, *then*

$$\max_{|z|=1}\left|p'(z)\right| \le n\frac{n|a_0|+\mu|a_\mu|k^{\mu+1}}{\left(1+k^{\mu+1}\right)n|a_0|+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)}\max_{|z|=1}\left|p(z)\right|$$
$$-\frac{n}{k^n}\left\{1-\frac{n|a_0|+\mu|a_\mu|k^{\mu+1}}{\left(1+k^{\mu+1}\right)n|a_0|+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\min_{|z|=k}\left|p(z)\right|. \tag{2.1}$$

Inequality (2.1) *is sharp and equality holds for the polynomial* $p(z)=\left(z^\mu+k^\mu\right)^{\frac{n}{\mu}}$ *where n is a multiple of* μ *and* $1\le\mu<n$.

Lemma 2.2. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$, $1\le\mu\le n$, *is a polynomial of degree* n *such that* $p(z)\ne0$ *in* $|z|<k$, $k>0$, *then for* $0<r\le R\le k$,

$$\max_{|z|=R}\left|p(z)\right| \le \left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}\max_{|z|=r}\left|p(z)\right|-\left\{\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}-1\right\}\min_{|z|=k}\left|p(z)\right|. \tag{2.2}$$

Equality holds in (2.2) *for* $p(z)=\left(z^\mu+k^\mu\right)^{\frac{n}{\mu}}$ *where n is a multiple of* μ .

Lemma 2.2 was proved by Dewan et. al [5].

Lemma 2.3. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$ $,1\le\mu\le n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\ge1$, *then*

$$\mu|a_\mu|k^\mu\le n|a_0|. \tag{2.3}$$

Lemma 2.3 is due to Qazi [11, proof and Remark of Lemma 1].

Lemma 2.4. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$ $,1\le\mu\le n$, *is a polynomial of degree n having no zero in* $|z|<k,k>0$, *then the function*

$$f(t)=\frac{\left(n|a_0|t+\mu|a_\mu|k^{\mu+1}\right)\left(k^\mu+t^\mu\right)}{\left(t^{\mu+1}+k^{\mu+1}\right)n|a_0|+\mu|a_\mu|\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)} \tag{2.4}$$

is a non-decreasing function of t in $(0,k]$.

Proof of Lemma 2.4. We prove this by derivative test. Now, we have

$$f'(t)=\frac{\left(n|a_0|-\mu|a_\mu|k^\mu\right)\left\{(k-t)t^{\mu-1}\mu k^\mu\left(n|a_0|t+\mu|a_\mu|k^{\mu+1}\right)+\left(n|a_0|+\mu|a_\mu|k^\mu\right)\left(k^{\mu+1}t^\mu+k^{2\mu+1}\right)\right\}}{\left\{\left(t^{\mu+1}+k^{\mu+1}\right)n|a_0|+\mu|a_\mu|\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)\right\}^2},$$

which is non-negative, since by (2.3) of Lemma 2.3, $\left(n|a_0|-\mu|a_\mu|k^\mu\right)\ge0$, and $t\le k$.

Lemma 2.5. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$ $,1\le\mu\le n$, *is a polynomial of degree n having no zero in* $|z|<k,k>0$, *then the function*

$$g(t)=\frac{\left(n|a_0|k^{\mu+1}+\mu|a_\mu|k^{2\mu}t\right)}{\left(t^{\mu+1}+k^{\mu+1}\right)n|a_0|+\mu|a_\mu|\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)}, \tag{2.5}$$

is a non-increasing function of t in $(0,k]$.

Proof of Lemma 2.5. We can also prove this by derivative test. Now, we have

$$g'(t)=\frac{\left(\mu|a_\mu|\right)^2k^{3\mu+1}t^\mu(1-\mu)-\left(n|a_0|\right)^2k^{\mu+1}t^\mu(1+\mu)-n|a_0|k^{2\mu}\mu|a_\mu|t^{\mu-1}\left(k^2+\mu t^2\right)}{\left\{\left(t^{\mu+1}+k^{\mu+1}\right)n|a_0|+\mu|a_\mu|\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)\right\}^2}\le0,$$

as $1\le\mu\le n$.

Lemma 2.6. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon$ $,1\le\mu\le n$, *is a polynomial of degree n having no zero in* $|z|<k,k>0$, *then for* $0<r\le\rho\le k$,

$$M(p,\rho)\le\left(\frac{\rho^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}\left[1-S\left\{1-\left(\frac{r^\mu+k^\mu}{\rho^\mu+k^\mu}\right)^{\frac{n}{\mu}}\right\}\right]M(p,r)-\frac{nr^{n-1}}{k^n}\min_{|z|=k}|p(z)|B_\rho(\rho-r)-I_1. \tag{2.6}$$

where

$I_1 = n\min\limits_{|z|=k}|p(z)|\int\limits_r^\rho A_t\left\{\left(\frac{t^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}-1\right\}dt$ and S, B_ρ, A_t are as defined in the Theorem.

Inequality (2.6) *is best possible for* $p(z)=\left(z^\mu+k^\mu\right)^{\frac{n}{\mu}}$, *where n is a multiple of* μ.

Remark 1.5. If n is a multiple of μ, then for $0<r<\rho\le k$, we have

$$1-\left(\frac{r^\mu+k^\mu}{\rho^\mu+k^\mu}\right)^{\frac{n}{\mu}}=\frac{\left(\rho^\mu-r^\mu\right)}{\left(\rho^\mu+k^\mu\right)\left\{1-\left(\frac{r^\mu+k^\mu}{\rho^\mu+k^\mu}\right)\right\}}\left\{1-\left(\frac{r^\mu+k^\mu}{\rho^\mu+k^\mu}\right)^{\frac{n}{\mu}}\right\}$$

$$=\left(\frac{\rho^\mu-r^\mu}{k^\mu+\rho^\mu}\right)\left\{\left(\frac{r^\mu+k^\mu}{\rho^\mu+k^\mu}\right)^{\frac{n}{\mu}-1}+\left(\frac{r^\mu+k^\mu}{\rho^\mu+k^\mu}\right)^{\frac{n}{\mu}-2}+............+\left(\frac{r^\mu+k^\mu}{\rho^\mu+k^\mu}\right)+1\right\}$$

$$\ge\left(\frac{\rho^\mu-r^\mu}{k^\mu+\rho^\mu}\right)\frac{n}{\mu}\left(\frac{r^\mu+k^\mu}{\rho^\mu+k^\mu}\right)^{\frac{n}{\mu}-1}. \tag{2.7}$$

Also, for $r=\rho$, inequality (2.6) holds trivially and hence inequality (2.7) is true for $0<r\le\rho\le k$.

Proof of Lemma 2.6. Since $p(z)$ has no zero in $|z|<k$, $k>0$, the polynomial $P(z)=p(tz)$ where $0<t\le k$ has no zero in $|z|<\frac{k}{t}$, where $\frac{k}{t}\ge 1$. Hence applying *(*2.1) of Lemma 2.1 to the polynomial $P(z)$, we get

$$\max_{|z|=1}|P'(z)|\le n\left[\frac{n|a_0|+\mu|a_\mu t^\mu|\left(\frac{k}{t}\right)^{\mu+1}}{\left\{1+\left(\frac{k}{t}\right)^{\mu+1}\right\}n|a_0|+\mu|a_\mu t^\mu|\left\{\left(\frac{k}{t}\right)^{\mu+1}+\left(\frac{k}{t}\right)^{2\mu}\right\}}\right]\max_{|z|=1}|P(z)|$$

$$-\frac{n}{\left(\frac{k}{t}\right)^n}\left[\frac{1}{t}-\frac{1}{t}\frac{n|a_0|+\mu|a_\mu t^\mu|\left(\frac{k}{t}\right)^{\mu+1}}{\left\{1+\left(\frac{k}{t}\right)^{\mu+1}\right\}n|a_0|+\mu|a_\mu t^\mu|\left\{\left(\frac{k}{t}\right)^{\mu+1}+\left(\frac{k}{t}\right)^{2\mu}\right\}}\right]\min_{|z|=\frac{k}{t}}|P(z)|$$

which implies

$$\max_{|z|=t}|p'(z)|\le nA_t\max_{|z|=t}|p(z)|-\frac{n}{k^n}B_t\min_{|z|=k}|p(z)|. \tag{2.8}$$

where $$B_t=\frac{n|a_0|k^{\mu+1}+\mu|a_\mu|k^{2\mu}t}{\left(t^{\mu+1}+k^{\mu+1}\right)n|a_0|+\mu|a_\mu|\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)}.$$

Now, for $0 < r \le \rho \le k$ and $0 \le \theta \le 2\pi$, we have

$$\left|p\left(\rho e^{i\theta}\right)-p\left(re^{i\theta}\right)\right| \le \int_r^\rho \left|p'\left(te^{i\theta}\right)\right|dt \le \int_r^\rho nA_t \max_{|z|=t}\left|p(z)\right|dt - \frac{n}{k^n}\min_{|z|=k}\left|p(z)\right|\int_r^\rho B_t\,dt, \quad \text{(by (2.8))}$$

which implies on using inequality (2.2) of Lemma 2.2,

$$\left|p\left(\rho e^{i\theta}\right)-p\left(re^{i\theta}\right)\right| \le \int_r^\rho nA_t\left[\left(\frac{t^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}\max_{|z|=r}\left|p(z)\right|-\left\{\left(\frac{t^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}-1\right\}\min_{|z|=k}\left|p(z)\right|\right]dt$$

$$-\frac{n}{k^n}\min_{|z|=k}\left|p(z)\right|\int_r^\rho B_t t^{n-1}\,dt$$

which gives for $0 < r \le \rho \le k$

$$M(p,\rho) \le 1+\frac{n}{\left(r^\mu+k^\mu\right)^{\frac{n}{\mu}}}M(p,r)\int_r^R A_t\left(t^\mu+k^\mu\right)^{\frac{n}{\mu}}dt$$

$$-\int_r^\rho nA_t\left\{\left(\frac{t^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}-1\right\}\min_{|z|=k}\left|p(z)\right|dt-\frac{n}{k^n}\min_{|z|=k}\left|p(z)\right|\int_r^\rho B_t t^{n-1}\,dt. \tag{2.9}$$

For $0 < r \le t \le \rho \le k$, by Lemma 2.4, we have

$$\frac{\left(n|a_0|t+\mu|a_\mu|k^{\mu+1}\right)\left(t^\mu+k^\mu\right)}{\left(t^{\mu+1}+k^{\mu+1}\right)n|a_0|+\mu|a_\mu|\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)} \le \frac{\left(n|a_0|\rho+\mu|a_\mu|k^{\mu+1}\right)\left(\rho^\mu+k^\mu\right)}{\left(\rho^{\mu+1}+k^{\mu+1}\right)n|a_0|+\mu|a_\mu|\left(k^{\mu+1}\rho^\mu+k^{2\mu}\rho\right)}. \tag{2.10}$$

Combining (2.10) with (2.9), we get

$$M(p,\rho) \le \left[1+\frac{\left(\rho^\mu+k^\mu\right)}{\left(r^\mu+k^\mu\right)^{\frac{n}{\mu}}}\left\{\frac{\left(n|a_0|\rho+\mu|a_\mu|k^{\mu+1}\right)}{\left(\rho^{\mu+1}+k^{\mu+1}\right)n|a_0|+\mu|a_\mu|\left(k^{\mu+1}\rho^\mu+k^{2\mu}\rho\right)}\right\}\right.$$

$$\left.\times\frac{n}{\mu}\int_r^\rho\left(t^\mu+k^\mu\right)^{\frac{n}{\mu}-1}\mu t^{\mu-1}dt\right]M(p,r)-I_1-I_2. \tag{2.11}$$

where

$$I_1 = n\min_{|z|=k}\left|p(z)\right|\int_r^\rho A_t\left\{\left(\frac{t^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}-1\right\}dt \quad\text{and}\quad I_2=\frac{n}{k^n}\min_{|z|=k}\left|p(z)\right|\int_r^\rho B_t t^{n-1}\,dt.$$

Now

$$I_2 \ge \frac{n}{k^n}\min_{|z|=k}\left|p(z)\right|r^{n-1}\int_r^\rho B_t\,dt. \tag{2.12}$$

If we apply Lemma 2.5 to the integrand of (2.12), we obtain

$$I_2 \ge \frac{nr^{n-1}}{k^n}\min_{|z|=k}\left|p(z)\right|B_\rho(\rho-r) \text{ with } B_\rho \text{ as defined in the Theorem.} \tag{2.13}$$

Using (2.13) to (2.11), we have

$$M(p,\rho)\le\left[1-\left\{\frac{(\rho^{\mu}+k^{\mu})(n|a_0|\rho+\mu|a_\mu|k^{\mu+1})}{(\rho^{\mu+1}+k^{\mu+1})n|a_0|+\mu|a_\mu|(k^{\mu+1}\rho^{\mu}+k^{2\mu}\rho)}\right\}\right.$$
$$\left.+\left\{\frac{(\rho^{\mu}+k^{\mu})(n|a_0|\rho+\mu|a_\mu|k^{\mu+1})}{(\rho^{\mu+1}+k^{\mu+1})n|a_0|+\mu|a_\mu|(k^{\mu+1}\rho^{\mu}+k^{2\mu}\rho)}\right\}\left(\frac{\rho^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}\right]M(p,r)$$
$$-\frac{nr^{n-1}}{k^n}\min_{|z|=k}|p(z)|B_\rho(\rho-r)-I_1.$$
$$=\left\{S+(1-S)\left(\frac{\rho^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}\right\}M(p,r)\;-\frac{nr^{n-1}}{k^n}\min_{|z|=k}|p(z)|B_\rho(\rho-r)-I_1$$
$$=\left(\frac{\rho^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}\left[1-S\left\{1-\left(\frac{r^{\mu}+k^{\mu}}{\rho^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}\right\}\right]M(p,r)\;-\frac{nr^{n-1}}{k^n}\min_{|z|=k}|p(z)|B_\rho(\rho-r)-I_1,$$

which completes the proof of Lemma 2.6.

Lemma 2.7. *If* $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon, 1\le\mu\le n$, *is a polynomial of degree n having no zero in the disc* $|z|<k, k\ge 1$, *then*

$$\max_{|z|=1}|p'(z)|\le\frac{n}{1+k^{\mu}}\left\{\max_{|z|=1}|p(z)|-\min_{|z|=k}|p(z)|\right\}. \tag{2.12}$$

The result is best possible for polynomial is $p(z)=(z^{\mu}+k^{\mu})^{\frac{n}{\mu}}$ *where n is a multiple of* μ.

Pukhta [9, Theorem 1.4], (see also [1] and [6]) obtained this result.

III. PROOF OF THE THEOREM

Since $p(z)=a_0+\sum_{\upsilon=\mu}^{n}a_\upsilon z^\upsilon, 1\le\mu\le n$, *is* has no zero in $|z|<k, k>0$, then for $0<\rho\le k$, $P(z)=p(\rho z)$ has no zero in $|z|<\frac{k}{\rho},\frac{k}{\rho}\ge 1$. Thus on using Lemma 2.6, we have

$$\max_{|z|=1}|P'(z)|\le\frac{n}{1+\left(\frac{k}{\rho}\right)^{\mu}}\left\{\max_{|z|=1}|P(z)|-\min_{|z|=\frac{k}{t}}|P(z)|\right\},$$

where

$$m=\min_{|z|=\frac{k}{\rho}}|P(z)|=\min_{|z|=\frac{k}{\rho}}|p(tz)|=\min_{|z|=k}|p(z)|,$$

which is equivalent to

$$\max_{|z|=\rho}|p'(z)|\le\frac{n\rho^{n-1}}{\rho^{\mu}+k^{\mu}}\left\{\max_{|z|=\rho}|p(z)|-\min_{|z|=k}|p(z)|\right\}. \tag{3.1}$$

Now for $0 < r \le \rho \le k$, applying inequality (2.5) of Lemma 2.5 to (3.1), we have

$$\max_{|z|=\rho}|p'(z)| \le \frac{n\rho^{n-1}}{\rho^{\mu}+k^{\mu}}\left(\frac{\rho^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}\left[1-S\left\{1-\left(\frac{r^{\mu}+k^{\mu}}{\rho^{\mu}+k^{\mu}}\right)^{\frac{n}{\mu}}\right\}\right]M(p,r)$$

$$-\frac{n\rho^{n-1}}{\rho^{\mu}+k^{\mu}}\left[I_1+\left\{\frac{nr^{n-1}}{k^n}B_{\rho}+1\right\}\min_{|z|=k}|p(z)|\right],$$

which completes the proof of the Theorem.

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