On Hyperbolic Hsu-structure Metric Manifold

[1]Prashant Kumar Shukla , [2]Geeta Verma , [3]Virendra Nath Pathak

[1] Department of Mathematics
Shri Ramswaroop Memorial Group Of Professional Colleges, Lucknow
pshukla7@yahoo.com

[2] Department of Mathematics
Shri Ramswaroop Memorial Group Of Professional Colleges, Lucknow
geeta_verma153@rediffmail.com

[3] Department of Mathematics
Shri Ramswaroop Memorial University Uttar Pradesh
pathak_virendra@rediffmail.com

Abstract: In this paper we have proved theorems of different kinds in n-recurrent and n-recurrent symmetric hyperbolic Hsu-structure metric manifolds involving equivalent conditions with respect to associated Pseudo H-Projective curvature tensor, associated Pseudo Bochner curvature tensor and Ricci tensor.

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1.Introduction

Let us consider a differentiable manifold

Consider a differentiable manifold M_n of differentiability class C^∞ [1]. Let there exist in M_n a vector valued linear function F of class C^∞ [2], satisfying the algebraic equation[8]:

(1.1a) $\overline{\overline{X}} = a^r X$, for arbitrary vector field X,

where

(1.1b) $\overline{X} \overset{def}{=} FX$ and'a' is a complex number.

Then {F}is said to give to M_n a hyperbolic differentiable structure, briefly known as HHsu-structure, defined by the equations (1.1) and the manifold M_n is called HHsu-manifold. The equations (1.1) give different structures for different values of 'a'[13].

Definition: A structure on an n-dimensional manifold M of class C^{∞} given by a non-null tensor field F satisfying

$$F^2 = a^r I$$

is called π - structure or Hsu-structure, where a is a non zero complex constant and I denotes the unit tensor field. Then M is called π - structure manifold or Hsu-structure manifold[7].

If $a^{r/2} \neq 0,$ it is Hyperbolic $\pi - structure,$ If $a^{r/2} = \pm 1,$ it is an almost complex or an almost hyperbolic Product structure, if $a^{r/2} = \pm i,$ it is an almost Product or almost hyperbolic Complex structure and if $a^{r/2} = 0,$ it is an almost tangent or hyperbolic almost tangent structure.

Let the HHsu-structure be endowed with a hermite tensor g, such that

(1.2) $$g(\overline{X},\overline{Y}) - a^r g(X,Y) = 0.$$

Then{F,g}is said to give to M_n, a hyperbolic differentiable metric structure and the manifold M_n is called a hyperbolic differentiable metric structure manifold.

Let us put

(1.3) $$'F(X, Y) \overset{def}{=} g(\overline{X}, Y).$$

Then the following equations hold:

(1.4a) $$'F(X, Y) = -'F(Y, X),$$

i.e. $'F$ is skew-symmetric in X and Y.

(1.4b) $$'F(\overline{X}, \overline{Y}) = a^r\, 'F(X, Y),$$

i.e. $'F$ is hybrid in X and Y.

(1.4c) $$'F(\overline{X}, Y) = -'F(X, \overline{Y}).$$

A bilinear function B in HHsu-metric manifold is said to be pure in the two slots, if

(1.5) $$B(\overline{X}, \overline{Y}) + a^r B(X, Y) = 0.$$

It is said to be hybrid in the two slots, if

(1.6) $$B(\overline{X}, \overline{Y}) - a^r B(X, Y) = 0.$$

Let D be a connexion and X, Y and Z be C^{∞} vector fields in M_n, then the function K,

defined by

(1.7) $$K(X, Y, Z) \overset{def}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z,$$

is called the curvature tensor of the connexion D.

Let us put

(1.8) $${}^{'}K(X, Y, Z, T) \overset{def}{=} g(K(X, Y, Z), T).$$

Then ${}^{'}K$ is real-valued 4-linear function, called associated curvature tensor or Riemann christoffel curvature tensor of the firstkind which satisfies the following properties:

(i) It is skew-symmetric in first two slots:

(1.9) $${}^{'}K(X, Y, Z, T) = -{}^{'}K(Y, X, Z, T).$$

(ii) It is skew-symmetric in last two slots:

(1.10) $${}^{'}K(X, Y, Z, T) = -{}^{'}K(X, Y, T, Z).$$

(iii) It is symmetric in two pairs of slots:

(1.11) $${}^{'}K(X, Y, Z, T) = {}^{'}K(Z, T, X, Y).$$

(iv) It satisfies Bianchi's first identities:

(1.12) $${}^{'}K(X, Y, Z, T) + {}^{'}K(Y, Z, X, T) + {}^{'}K(Z, X, Y, T) = 0$$

(v) It also satisfies Bianchi's second identities:

(1.13) $$(D_X {}^{'}K)(Y, Z, T, U) + (D_Y {}^{'}K)(Z, X, T, U) + (D_Z {}^{'}K)(X, Y, T, U) = 0.$$

The tensor defined by

(1.14) $$Ric(Y, Z) \overset{def}{=} (C_1^1 K)(Y, Z),$$

is called Ricci tensor, C_1^1 being the contraction operator[5].

It is a symmetric tensor of the type (0, 2):

(1.15) $$Ric(Y, Z) = Ric(Z, Y)$$

The linear map r, defined by

(1.16) $g(r(X),Y)=g(X,r(Y))$ is called Ricci map.

The scalar R, defined by

(1.17) $R=(C_1^1 r)$ is called the scalar curvature of M_n at any point p[6].

In the HHsu-metric structure manifold, Pseudo H-Conharmonic Curvature tensor S^*, and Pseudo Bochner Curvature tensor B^*, are given by

(1.18) $S^*(X,Y,Z)=K(X,Y,Z)-\frac{a^r}{(n+4)}[a^r Ric(Y,Z)X-a^r Ric(X,Z)Y-g(X,Z)r(Y)$

$+g(Y,Z)r(X)+Ric(X,\bar{Z})\bar{Y}-Ric(Y,\bar{Z})\bar{X}-2Ric(\bar{X},Y)\bar{Z}$

$+g(X,\bar{Z})r(\bar{Y})-g(Y,\bar{Z})r(\bar{X})-2g(\bar{X},Y)r(\bar{Z})]$,

(1.19) $B^*(X,Y,Z)=K(X,Y,Z)-\frac{a^r}{(n+4)}[a^r Ric(Y,Z)X-a^r Ric(X,Z)Y-g(X,Z)r(Y)$

$+g(Y,Z)r(X)+Ric(X,\bar{Z})\bar{Y}-Ric(Y,\bar{Z})\bar{X}-2Ric(\bar{X},Y)\bar{Z}$

$+g(X,\bar{Z})r(\bar{Y})-g(Y,\bar{Z})r(\bar{X})-2g(\bar{X},Y)r(\bar{Z})]$

$+\frac{a^r R}{(n+2)(n+4)}\left[g(Y,Z)X-g(X,Z)Y+g(X,\bar{Z})\bar{Y}-g(Y,\bar{Z})\bar{X}-2g(\bar{X},Y)\bar{Z}\right]$.

The associated Pseudo H-Conharmonic curvature tensor $'S^*$ and associated Pseudo Bochnercurvarure tensor $'B^*$ are given by[3]:

(1.20a) $'S^*(X,Y,Z,T)\overset{def}{=}g(S^*(X,Y,Z),T)$,

(1.20b) $'B^*(X,Y,Z,T)\overset{def}{=}g(B^*(X,Y,Z),T)$,

Consequently,

(1.21) $'S^*(X,Y,Z,T)='K(X,Y,Z,T)-\frac{a^r}{(n+4)}[a^r Ric(Y,Z)g(X,T)-a^r Ric(X,Z)g(Y,T)$

$$+Ric(X,\bar{Z})g(\bar{Y},T)-Ric(Y,\bar{Z})g(\bar{X},T)-2Ric(\bar{X},Y)g(\bar{Z},T)-g(X,Z)Ric(Y,T)$$

$$+g(Y,Z)Ric(X,T)-g(Y,\bar{Z})Ric(\bar{X},T)-2g(\bar{X},Y)Ric(\bar{Z},T)],$$

$$(1.22)\ 'B^{*}(X,Y,Z,T)='K(X,Y,Z,T)-\frac{a^{r}}{(n+4)}[a^{r}Ric(Y,Z)g(X,T)-a^{r}Ric(X,Z)g(Y,T)$$

$$-g(Y,Z)Ric(Y,T)+g(Y,Z)Ric(X,T)+Ric(X,\bar{Z})g(\bar{Y},T)$$

$$-Ric(Y,\bar{Z})g(\bar{X},T)-2Ric(\bar{X},Y)g(\bar{Z},T)+g(X,\bar{Z})Ric(\bar{Y},T)$$

$$-g(Y,\bar{Z})Ric(\bar{X},T)-2g(\bar{X},Y)Ric(\bar{Z},T)]+\frac{a^{r}R}{(n+2)(n+4)}[g(Y,Z)g(X,T)$$

$$-g(X,Z)g(Y,T)+g(X,\bar{Z})g(\bar{Y},T)-g(Y,\bar{Z})g(\bar{X},T)-2g(\bar{X},Y)g(\bar{Z},T)].$$

2. n-Recurrence and n-Recurrence Symmetry of Different Kinds

Theorem 2.1. *In the hyperbolic general structure metric manifold* $V_{n,}$ *if any two of the following conditions hold for the same n-recurrence parameter then the third also holds[9]:*

(i) It is associated Pseudo H-Conharmonic (1) n-recurrent,
(ii) It is associated Pseudo Bochner (1) n-recurrent,
(iii) It is Ricci n-recurrent,

Provided

$$(2.1)\ \frac{a^{r}}{(n+2)(n+4)}[(\nabla_{n-1}....\nabla_{1}R)(U_{1},....,U_{n-1})\{g(Y,Z)\,g((\nabla_{n}F)(\bar{X},U_{n}),T)$$

$$-g((\nabla_{n}F)(\bar{X},U_{n}),Z)\,g(Y,T)+g((\nabla_{n}F)(\bar{X},U_{n}),\bar{Z})\,g(\bar{Y},T)$$

$$+a^{r}g(Y,\bar{Z})g((\nabla_{n}\bar{X})(U_{n}),T)+2a^{r}g((\nabla_{n}\bar{X})(U_{n}),Y)g(\bar{Z},T)\}$$

$$+(\nabla_{n}\nabla_{n-2}.....\nabla_{1}R)(U_{1},....,U_{n-2},U_{n})\{g(Y,Z)g((\nabla_{n-1}F)(\bar{X},U_{n-1}),T)$$

$$-g((\nabla_{n-1}F)(\bar{X},U_{n-1}),Z)\,g(Y,T)+g((\nabla_{n-1}F)(\bar{X},U_{n-1}),\bar{Z})\,g(\bar{Y},T)$$

$$+a^{r}g(Y,\bar{Z})g((\nabla_{n-1}\bar{X})(U_{n-1}),T)+2a^{r}g((\nabla_{n-1}\bar{X})(U_{n-1}),Y)g(\bar{Z},T)\}$$

$$......................+(\nabla_n....\nabla_3\nabla_1 R)(U_1,U_3,....,U_n)\{g(Y,Z)\,g((\nabla_2 F)(\bar{X},U_2),T)$$

$$-g((\nabla_2 F)(\bar{X},U_2),Z)\,g(Y,T)+g((\nabla_2 F)(\bar{X},U_2),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_2\bar{X})(U_2),T)+2a^r g((\nabla_2\bar{X})(U_2),Y)g(\bar{Z},T)\}$$

$$+(\nabla_n....\nabla_2 R)(U_2,...,U_n)\{g(Y,Z)g((\nabla_1 F)(\bar{X},U_1),T)$$

$$-g((\nabla_1 F)(\bar{X},U_1),Z)\,g(Y,T)+g((\nabla_1 F)(\bar{X},U_1),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_1\bar{X})(U_1),T)+2a^r g((\nabla_1\bar{X})(U_1),Y)g(\bar{Z},T)\}$$

$$+(\nabla_{n-2}....\nabla_1 R)(U_1,...,U_{n-2})\{g(Y,Z)g((\nabla_n\nabla_{n-1}F)(\bar{X},U_{n-1},U_n),T)$$

$$-g((\nabla_n\nabla_{n-1}F)(\bar{X},U_{n-1},U_n),Z)\,g(Y,T)+g((\nabla_n\nabla_{n-1}F)(\bar{X},\nabla_{n-1},U_n),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_n\nabla_{n-1}\bar{X})(U_{n-1},U_n),T)+2a^r g((\nabla_n\nabla_{n-1}\bar{X})(U_{n-1},U_n),Y)g(\bar{Z},T)\}$$

$$......................+(\nabla_n....\nabla_3 R)(U_3,....,U_n)\{g(Y,Z)\,g((\nabla_2\nabla_1 F)(\bar{X},U_1,U_2),T)$$

$$-g((\nabla_2\nabla_1 F)(\bar{X},U_1,U_2),Z)\,g(Y,T)+g((\nabla_2\nabla_1 F)(\bar{X},U_1,U_2),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_2\nabla_1\bar{X})(U_1,U_2),T)+2a^r g((\nabla_2\nabla_1\bar{X})(U_1,U_2),Y)g(\bar{Z},T)\}$$

$$......................+(\nabla_n R)(U_n)\{g(Y,Z)\,g((\nabla_{n-1}.....\nabla_1 F)(\bar{X},U_1,...,U_{n-1}),T)$$

$$-g((\nabla_{n-1}....\nabla_1 F)(\bar{X},U_1,..,U_{n-1}),Z)\,g(Y,T)+g((\nabla_{n-1}...\nabla_1 F)(\bar{X},U_1,..,U_{n-1}),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_{n-1}...\nabla_1\bar{X})(U_1,..,U_{n-1}),T)+2a^r g((\nabla_{n-1}...\nabla_1\bar{X})(U_1,...,U_{n-1}),Y)g(\bar{Z},T)\}$$

$$+(\nabla_1 R)(U_1)\{g(Y,Z)g((\nabla_n...\nabla_2 F)(\bar{X},U_2,..,U_n),T)-g((\nabla_n...\nabla_2 F)(\bar{X},U_2,..,U_n),Z)g(Y,T)$$

$$+g((\nabla_n....\nabla_2 F)(\bar{X},U_2,..,U_n),\bar{Z})\,g(\bar{Y},T)+a^r g(Y,\bar{Z})g((\nabla_n...\nabla_2\bar{X})(U_2,..,U_n),T)$$

$$+2a^r g((\nabla_n...\nabla_2\bar{X})(U_2,...,U_n),Y)g(\bar{Z},T)\}+R\{g(Y,Z)g((\nabla_n...\nabla_1 F)(\bar{X},U_1,..,U_n),T)$$

$$-g((\nabla_n....\nabla_1 F)(\bar{X},U_1,..,U_n),Z)\,g(Y,T)+g((\nabla_n...\nabla_1 F)(\bar{X},U_1,..,U_n),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_n...\nabla_1\bar{X})(U_1,..,U_n),T)+2a^r g((\nabla_n...\nabla_1\bar{X})(U_1,...,U_n),Y)g(\bar{Z},T)\}$$

$$=0.$$

Proof: From the equations (1.21) and (1.22), we have

$$(2.2)\ 'S^*(X,Y,Z,T) = 'B^*(X,Y,Z,T) - \frac{a^r R}{(n+2)(n+4)}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)$$

$$+ g(X,\bar{Z})g(\bar{Y},T) - g(Y,\bar{Z})g(\bar{X},T) - 2g(\bar{X},Y)g(\bar{Z},T)].$$

Barring X in equation (2.2) and using the equation (1.1a) in the resulting equation[10], we get

$$(2.3)\qquad 'S^*(\bar{X},Y,Z,T) = 'B^*(\bar{X},Y,Z,T) - \frac{a^r R}{(n+2)(n+4)}[g(Y,Z)g(\bar{X},T) - g(\bar{X},Z)g(Y,T)$$

$$- g(\bar{X},\bar{Z})g(\bar{Y},T) + a^r g(Y,\bar{Z})g(X,T) + 2a^r g(X,Y)g(\bar{Z},T)].$$

Multiplying the equation (2.3) by $B_n(U_1,............,U_n)$,barring X then using the equation (1.1a)in the resulting equation, we get

$$(2.4)\ \ a^r B_n(U_1,..,U_n)'S^*(X,Y,Z,T) = a^r B_n(U_1,..,U_n)'B^*(X,Y,Z,T)$$

$$- \frac{B_n(U_1,........,U_n)R}{(n+2)(n+4)}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)$$

$$+ g(X,\bar{Z})g(\bar{Y},T) - g(Y,\bar{Z})g(\bar{X},T) - 2g(\bar{X},Y)g(\bar{Z},T)].$$

Differentiating the equation (2.3) with respect to $U_1,...,U_n$, using the equation (2.3), then barring X and subtracting the equation (2.4) from the resulting equation[15], we get

$$(2.5)\ \ a^r(\nabla_n...\nabla_1 'S^*)(X,Y,Z,T,U_1,....U_n)$$

$$- (\nabla_{n-1}...\nabla_1 'S^*)((\nabla_n F)(\bar{X},U_n),Y,Z,T,U_1,....,U_{n-1}).............$$

$$- (\nabla_n...\nabla_3\nabla_1 'S^*)((\nabla_2 F)(\bar{X},U_2),Y,Z,T,U_1,U_3,....,U_n).............$$

$$- (\nabla_n...\nabla_3\nabla_2 'S^*)((\nabla_1 F)(\bar{X},U_1),Y,Z,T,U_2,U_3,....,U_n).............$$

$$- (\nabla_{n-2}...\nabla_1 'S^*)((\nabla_n\nabla_{n-1} F)(\bar{X},U_{n-1},U_n),Y,Z,T,U_1,....,U_{n-2}).............$$

$$-(\nabla_n ... \nabla_3 'S^*)((\nabla_2 \nabla_1 F)(\overline{X}, U_1, U_2), Y, Z, T, U_3, ..., U_n) ....$$

$$-(\nabla_n 'S^*)((\nabla_{n-1} ... \nabla_1 F)(\overline{X}, U_1, ..., U_{n-1}), Y, Z, T, U_n) ....$$

$$-(\nabla_1 'S^*)((\nabla_n ... \nabla_2 F)(\overline{X}, U_2, ..., U_n), Y, Z, T, U_1) ....$$

$$-'S^*((\nabla_n \nabla_{n-1} ... \nabla_1 F)(\overline{X}, U_1, ..., U_{n-1}, U_n), Y, Z, T) ....$$

$$= a^r ((\nabla_n ... \nabla_1 'B^*)(X, Y, Z, T, U_1, ..., U_n)$$

$$-(\nabla_{n-1} ... \nabla_1 'B^*)((\nabla_n F)(\overline{X}, U_n), Y, Z, T, U_1, ..., U_{n-1})$$

$$-(\nabla_n \nabla_{n-2} ... \nabla_1 'B^*)((\nabla_{n-1} F)(\overline{X}, U_{n-1}), Y, Z, T, U_1, ..., U_{n-2}, U_n) ....$$

$$-g((\nabla_2 F)(Y, U_2), \overline{Z})\, g(\overline{X}, (\nabla_1 F)(\overline{T}, U_1))$$

$$-(\nabla_n ... \nabla_3 \nabla_2 'B^*)((\nabla_1 F)(\overline{X}, U_1), Y, Z, T, U_2, U_3, ..., U_n)$$

$$-(\nabla_{n-2} ... \nabla_1 'B^*)((\nabla_n \nabla_{n-1} F)(\overline{X}, U_{n-1}, U_n), Y, Z, T, U_1, ..., U_{n-2}) ....$$

$$-(\nabla_n ... \nabla_3 'B^*)((\nabla_2 \nabla_1 F)(\overline{X}, U_1, U_2), Y, Z, T, U_3, ..., U_n) ....$$

$$-(\nabla_n 'B^*)((\nabla_{n-1} .... \nabla_1 F)(\overline{X}, U_1, ..., U_{n-1}), Y, Z, T, U_n) ....$$

$$-(\nabla_1 'B^*)((\nabla_n .... \nabla_2 F)(\overline{X}, U_2, ..., U_n), Y, Z, T, U_1) ....$$

$$-'B^*((\nabla_n \nabla_{n-1} .... \nabla_1 F)(\overline{X}, U_1, ..., U_{n-1}, U_n), Y, Z, T)$$

$$-a^r B_n(U_1, U_2, ... U_n)'B^*(X, Y, Z, T)$$

$$-\frac{1}{(n+2)(n+4)}\{(\nabla_n .... \nabla_1 R)(U_1, ..., U_n) - RB_n(U_1, ..., U_n)\}$$

$$[g(Y,Z)g(X,T) - g(X,Z)g(Y,T) + g(X,\overline{Z})g(\overline{Y},T) - g(Y,\overline{Z})g(\overline{X},T) - 2g(\overline{X},Y)g(\overline{Z},T)]$$

$$\frac{a^r}{(n+2)(n+4)}[(\nabla_{n-1} .... \nabla_1 R)(U_1, ..., U_{n-1})\{g(Y,Z)\, g((\nabla_n F)(\overline{X}, U_n), T)$$

$$-g((\nabla_n F)(\overline{X}, U_n), Z)\, g(Y,T) + g((\nabla_n F)(\overline{X}, U_n), \overline{Z})\, g(\overline{Y}, T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_n \bar{X})(U_n),T)+2a^r g((\nabla_n \bar{X})(U_n),Y)g(\bar{Z},T)\}$$

$$+(\nabla_n \nabla_{n-2}.....\nabla_1 R)(U_1,....,U_{n-2},U_n)\{g(Y,Z)g((\nabla_{n-1}F)(\bar{X},U_{n-1}),T)$$

$$-g((\nabla_{n-1}F)(\bar{X},U_{n-1}),Z)\,g(Y,T)+g((\nabla_{n-1}F)(\bar{X},U_{n-1}),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_{n-1}\bar{X})(U_{n-1}),T)+2a^r g((\nabla_{n-1}\bar{X})(U_{n-1}),Y)g(\bar{Z},T)\}$$

$$......................+(\nabla_n....\nabla_3\nabla_1 R)(U_1,U_3,....,U_n)\{g(Y,Z)\,g((\nabla_2 F)(\bar{X},U_2),T)$$

$$-g((\nabla_2 F)(\bar{X},U_2),Z)\,g(Y,T)+g((\nabla_2 F)(\bar{X},U_2),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_2 \bar{X})(U_2),T)+2a^r g((\nabla_2 \bar{X})(U_2),Y)g(\bar{Z},T)\}$$

$$+(\nabla_n....\nabla_2 R)(U_2,...,U_n)\{g(Y,Z)g((\nabla_1 F)(\bar{X},U_1),T)$$

$$-g((\nabla_1 F)(\bar{X},U_1),Z)\,g(Y,T)+g((\nabla_1 F)(\bar{X},U_1),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_1 \bar{X})(U_1),T)+2a^r g((\nabla_1 \bar{X})(U_1),Y)g(\bar{Z},T)\}$$

$$+(\nabla_{n-2}....\nabla_1 R)(U_1,...,U_{n-2})\{g(Y,Z)g((\nabla_n\nabla_{n-1}F)(\bar{X},U_{n-1},U_n),T)$$

$$-g((\nabla_n\nabla_{n-1}F)(\bar{X},U_{n-1},U_n),Z)\,g(Y,T)+g((\nabla_n\nabla_{n-1}F)(\bar{X},U_{n-1},U_n),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_n\nabla_{n-1}\bar{X})(U_{n-1},U_n),T)+2a^r g((\nabla_n\nabla_{n-1}\bar{X})(U_{n-1},U_n),Y)g(\bar{Z},T)\}$$

$$......................+(\nabla_n....\nabla_3 R)(U_3,....,U_n)\{g(Y,Z)\,g((\nabla_2\nabla_1 F)(\bar{X},U_1,U_2),T)$$

$$-g((\nabla_2\nabla_1 F)(\bar{X},U_1,U_2),Z)\,g(Y,T)+g((\nabla_2\nabla_1 F)(\bar{X},U_1,U_2),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_2\nabla_1\bar{X})(U_1,U_2),T)+2a^r g((\nabla_2\nabla_1\bar{X})(U_1,U_2),Y)g(\bar{Z},T)\}$$

$$......................+(\nabla_n R)(U_n)\{g(Y,Z)\,g((\nabla_{n-1}.....\nabla_1 F)(\bar{X},U_1,...,U_{n-1}),T)$$

$$-g((\nabla_{n-1}....\nabla_1 F)(\bar{X},U_1,..,U_{n-1}),Z)\,g(Y,T)+g((\nabla_{n-1}...\nabla_1 F)(\bar{X},U_1,..,U_{n-1}),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_{n-1}.....\nabla_1\bar{X})(U_1,......,U_{n-1}),T)+2a^r g((\nabla_{n-1}...\nabla_1\bar{X})(U_1,...,U_{n-1}),Y)g(\bar{Z},T)\}...........$$

$$+(\nabla_1 R)(U_1)\{g(Y,Z)g((\nabla_n...\nabla_2 F)(\bar{X},U_2,..,U_n),T)-g((\nabla_n...\nabla_2 F)(\bar{X},U_2,..,U_n),Z)g(Y,T)$$

$$+g((\nabla_n....\nabla_2 F)(\bar{X},U_2,..,U_n),\bar{Z})\,g(\bar{Y},T)+a^r g(Y,\bar{Z})g((\nabla_n...\nabla_2\bar{X})(U_2,..,U_n),T)$$

$$+2a^r g((\nabla_n ... \nabla_2 \overline{X})(U_2,...,U_n),Y)g(\overline{Z},T)\} + R\{g(Y,Z)g((\nabla_n ... \nabla_1 F)(\overline{X},U_1,..,U_n),T)$$

$$-g((\nabla_n .... \nabla_1 F)(\overline{X},U_1,..,U_n),Z)\, g(Y,T) + g((\nabla_n ... \nabla_1 F)(\overline{X},U_1,..,U_n),\overline{Z})\, g(\overline{Y},T)$$

$$+a^r g(Y,\overline{Z})g((\nabla_n ... \nabla_1 \overline{X})(U_1,..,U_n),T) + 2a^r g((\nabla_n ... \nabla_1 \overline{X})(U_1,...,U_n),Y)g(\overline{Z},T)\}$$

Let the hyperbolic general structure metric manifold is associated Pseudo H-Conharmonic (1) n-recurrent and associated Pseudo Bochner (1) n-recurrent for the same n-recurrence parameter and the equation (2.1) is satisfied[4], then from the equation (2.5), we have

$$(2.6)\ \frac{1}{(n+2)(n+4)}\{(\nabla_n ....\nabla_1 R)(U_1,...,U_n) - RB_n(U_1,....,U_n)\}[g(Y,Z)g(X,T)$$

$$-g(X,Z)g(Y,T) + g(X,\overline{Z})g(\overline{Y},T) - g(Y,\overline{Z})g(\overline{X},T) - 2g(\overline{X},Y)g(\overline{Z},T)] = 0$$

or

$$(\nabla_n ....\nabla_1 R)(U_1,.........,U_n) = RB_n(U_1,.................,U_n),$$

Which shows that the manifold is Ricci n-recurrent.

Similarly it can be shown that if the hyperbolic general structure metric manifold is either associated Pseudo H-Conharmonic (1) n-recurrent and Ricci n-recurrent or associated Pseudo Bochner (1) n-recurrent and Ricci n-recurrent then it is either associated Pseudo Bochner (1) n-recurrent or associated Pseudo H-Conharmonic (1) n-recurrent for the same n-recurrence parameter provided the equation (2.1) is satisfied.

Theorem 2.2.*In the hyperbolic general structure metric manifold M_n, if any two of the following conditions hold for the same n-recurrence parameter then the third also holds[12]:*

- *(i) It is associated Pseudo H-Conharmonic(1) n-recurrent symmetric,*
- *(ii) It is associated PseudoBochner (1) n-recurrent symmetric,*
- *(iii) It is Ricci n-recurrence symmetric,*

Provided

$$(2.7)\ \frac{a^r}{(n+2)(n+4)}[(\nabla_{n-1}....\nabla_1 R)(U_1,...., U_{n-1})\{g(Y,Z)\, g((\nabla_n F)(\overline{X},U_n),T)$$

$$-g((\nabla_n F)(\overline{X},U_n),Z)\, g(Y,T) + g((\nabla_n F)(\overline{X},U_n),\overline{Z})\, g(\overline{Y},T)$$

$$+a^r g(Y,\overline{Z})g((\nabla_n \overline{X})(U_n),T) + 2a^r g((\nabla_n \overline{X})(U_n),Y)g(\overline{Z},T)\}$$

$$+(\nabla_n\nabla_{n-2}.....\nabla_1 R)(U_1,....,U_{n-2},U_n)\{g(Y,Z)g((\nabla_{n-1}F)(\bar{X},U_{n-1}),T)$$

$$-g((\nabla_{n-1}F)(\bar{X},U_{n-1}),Z)\,g(Y,T)+g((\nabla_{n-1}F)(\bar{X},U_{n-1}),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_{n-1}\bar{X})(U_{n-1}),T)+2a^r g((\nabla_{n-1}\bar{X})(U_{n-1}),Y)g(\bar{Z},T)\}................$$

$$+(\nabla_n....\nabla_3\nabla_1 R)(U_1,U_3,....,U_n)\{g(Y,Z)\,g((\nabla_2 F)(\bar{X},U_2),T)$$

$$-g((\nabla_2 F)(\bar{X},U_2),Z)\,g(Y,T)+g((\nabla_2 F)(\bar{X},U_2),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_2\bar{X})(U_2),T)+2a^r g((\nabla_2\bar{X})(U_2),Y)g(\bar{Z},T)\}$$

$$+(\nabla_n....\nabla_2 R)(U_2,...,U_n)\{g(Y,Z)g((\nabla_1 F)(\bar{X},U_1),T)-g((\nabla_1 F)(\bar{X},U_1),Z)\,g(Y,T)$$

$$+g((\nabla_1 F)(\bar{X},U_1),\bar{Z})\,g(\bar{Y},T)+a^r g(Y,\bar{Z})g((\nabla_1\bar{X})(U_1),T)+2a^r g((\nabla_1\bar{X})(U_1),Y)g(\bar{Z},T)\}$$

$$+(\nabla_{n-2}....\nabla_1 R)(U_1,...,U_{n-2})\{g(Y,Z)g((\nabla_n\nabla_{n-1}F)(\bar{X},U_{n-1},U_n),T)$$

$$-g((\nabla_n\nabla_{n-1}F)(\bar{X},U_{n-1},U_n),Z)\,g(Y,T)+g((\nabla_n\nabla_{n-1}F)(\bar{X},U_{n-1},U_n),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_n\nabla_{n-1}\bar{X})(U_{n-1},U_n),T)+2a^r g((\nabla_n\nabla_{n-1}\bar{X})(U_{n-1},U_n),Y)g(\bar{Z},T)\}$$

$$......................+(\nabla_n....\nabla_3 R)(U_3,....,U_n)\{g(Y,Z)\,g((\nabla_2\nabla_1 F)(\bar{X},U_1,U_2),T)$$

$$-g((\nabla_2\nabla_1 F)(\bar{X},U_1,U_2),Z)\,g(Y,T)+g((\nabla_2\nabla_1 F)(\bar{X},U_1,U_2),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_2\nabla_1\bar{X})(U_1,U_2),T)+2a^r g((\nabla_2\nabla_1\bar{X})(U_1,U_2),Y)g(\bar{Z},T)\}$$

$$......................+(\nabla_n R)(U_n)\{g(Y,Z)\,g((\nabla_{n-1}.....\nabla_1 F)(\bar{X},U_1,...,U_{n-1}),T)$$

$$-g((\nabla_{n-1}....\nabla_1 F)(\bar{X},U_1,..,U_{n-1}),Z)\,g(Y,T)+g((\nabla_{n-1}...\nabla_1 F)(\bar{X},U_1,..,U_{n-1}),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_{n-1}.....\nabla_1\bar{X})(U_1,.......,U_{n-1}),T)+2a^r g((\nabla_{n-1}...\nabla_1\bar{X})(U_1,....,U_{n-1}),Y)g(\bar{Z},T)\}...........$$

$$+(\nabla_1 R)(U_1)\{g(Y,Z)g((\nabla_n...\nabla_2 F)(\bar{X},U_2,..,U_n),T)-g((\nabla_n...\nabla_2 F)(\bar{X},U_2,..,U_n),Z)g(Y,T)$$

$$+g((\nabla_n....\nabla_2 F)(\bar{X},U_2,..,U_n),\bar{Z})\,g(\bar{Y},T)+a^r g(Y,\bar{Z})g((\nabla_n...\nabla_2\bar{X})(U_2,..,U_n),T)$$

$$+2a^r g((\nabla_n...\nabla_2\bar{X})(U_2,...,U_n),Y)g(\bar{Z},T)\}+R\{g(Y,Z)g((\nabla_n...\nabla_1 F)(\bar{X},U_1,..,U_n),T)$$

$$-g((\nabla_n....\nabla_1 F)(\bar{X},U_1,..,U_n),Z)\,g(Y,T)+g((\nabla_n...\nabla_1 F)(\bar{X},U_1,..,U_n),\bar{Z})\,g(\bar{Y},T)$$

$$+a^r g(Y,\bar{Z})g((\nabla_n...\nabla_1\bar{X})(U_1,..,U_n),T)+2a^r g((\nabla_n...\nabla_1\bar{X})(U_1,...,U_n),Y)g(\bar{Z},T)\}=0.$$

Proof:Let the hyperbolic Hsu-structure metric manifold is associated Pseudo H-Conharmonic (1) n-recurrent symmetric and associated Pseudo Bochner (1) n-recurrent symmetric then from the equation (2.5), we have

(2.8) $$(\nabla_n....\nabla_1 R)(U_1,...,U_n)[g(Y,Z)g(X,T)-g(X,Z)g(Y,T)+g(X,\bar{Z})g(\bar{Y},T)$$

$$-g(Y,\bar{Z})g(\bar{X},T)-2g(\bar{X},Y)g(\bar{Z},T)]=0.$$

or

$$(\nabla_n................\nabla_1 R)(U_1,................,U_n)=0.$$

provided the equation (2.7) is satisfied, which shows that the manifold is Ricci n-recurrent symmetric[14].

Similarly, it can be shown that if the hyperbolic Hsu-structure metric manifold is either associated Pseudo H-Conharmonic (1) n-recurrent symmetric and Ricci n-recurrent symmetric or associated Pseudo Bochner (1) n-recurrent symmetric and Ricci n-recurrent symmetric or associated Pseudo H-Conharmonic (1) n-recurrent symmetric for the same n-recurrence parameter provided the equation (2.7) is satisfied.

Note 2.1. Theorems of the type (2.1) and (2.2) can also be statedand proved taking (2), (3), (4), (12), (13), (14), (23), (34), (123), (124), (134), (234), or (1234) n-recurrent and (2), (3), (4), (12), (13), (14), (23), (34), (123), (124), (134), (234), or (1234) n-recurrent symmetric hyperbolic Hsu-structure metric manifold.

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